2. Consider the ordered logit model with unobserved utility given by

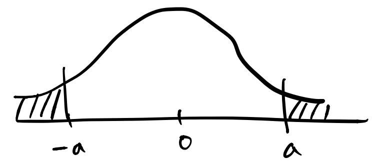
$$y_i^* = x_i'\beta + \epsilon_i \quad \epsilon_i | x_i \overset{i.i.d.}{\sim} Logistic(0,1)$$

$\Lambda(\cdot)$: CDF

$P(\epsilon_i < \cdot \mid x_i) = F_{\epsilon|x}(\cdot) \equiv \Lambda(\cdot)$

We observe x_i and y_i, where

$$\begin{cases} y_i = 0 \text{ if } y_i^* < \gamma_1 \\ y_i = 1 \text{ if } \gamma_1 \le y_i^* < \gamma_2 \\ y_i = 2 \text{ if } \gamma_2 \le y_i^* < \gamma_3 \\ y_i = 3 \text{ if } y_i^* \ge \gamma_3 \end{cases}$$



Our goal is to estimate β, γ_1, γ_2, and γ_3.

① Conditional probabilities

$P(y_i=0 \mid x_i) = P(y_i^* < \gamma_1 \mid x_i) = P(x_i'\beta + \epsilon_i < \gamma_1 \mid x_i) = P(\epsilon_i < \gamma_1 - x_i'\beta \mid x_i)$

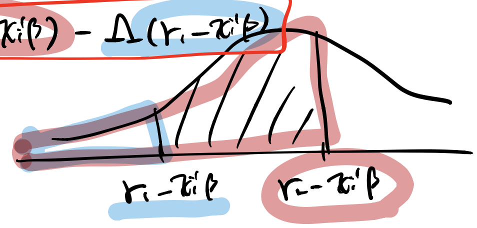
$\overset{①}{=} \Lambda(\gamma_1 - x_i'\beta)$.

$P(y_i=1 \mid x_i) = P(\gamma_1 \le y_i^* < \gamma_2 \mid x_i) = P(\gamma_1 \le x_i'\beta + \epsilon_i < \gamma_2 \mid x_i)$

$= P(\gamma_1 - x_i'\beta \le \epsilon_i < \gamma_2 - x_i'\beta \mid x_i)$

$\overset{②}{=} \boxed{\Lambda(\gamma_2 - x_i'\beta) - \Lambda(\gamma_1 - x_i'\beta)}$

$P(y_i=2 \mid x_i) \overset{③}{=} \boxed{\Lambda(\gamma_3 - x_i'\beta) - \Lambda(\gamma_2 - x_i'\beta)}$

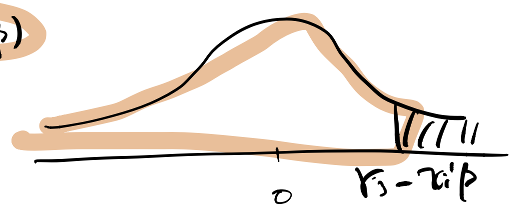


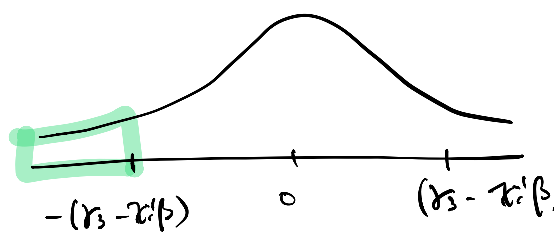
$P(y_i=3 \mid x_i) = P(y_i^* \ge \gamma_3)$

$= P(\epsilon \ge \gamma_3 - x_i'\beta \mid x_i)$

$= 1 - \Lambda(\gamma_3 - x_i'\beta)$

$\overset{④}{=} \boxed{\Lambda(x_i'\beta - \gamma_3)}$





Step 2: Likelihood fn.

$= \begin{cases} 1 & \text{if } y_i = 0 \\ 0 & \text{if } y_i \ne 0 \end{cases}$

$L(\gamma_1, \gamma_2, \gamma_3, \beta) \equiv \prod_{i=1}^{n} P(y_i=0 \mid x_i)^{1(y_i=0)} P(y_i=1 \mid x_i)^{1(y_i=1)} P(y_i=2 \mid x_i)^{1(y_i=2)} P(y_i=3 \mid x_i)^{1(y_i=3)}$

$= \prod_{y_i=0} P(y_i=0 \mid x_i) \prod_{y_i=1} P(y_i=1 \mid x_i) \prod_{y_i=2} P(y_i=2 \mid x_i) \prod_{y_i=3} P(y_i=3 \mid x_i)$

$= \prod_{y_i=0} [\Lambda(\gamma_1 - x_i'\beta)] \prod_{y_i=1} [\ ② \ ] \prod_{y_i=2} [\ ③ \ ] \prod_{y_i=3} [\ ④ \ ]$

(b) A.M.E.

recall
$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

By def.

① $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial P(y_i=0|x_i)}{\partial x_i}\right] = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial \Lambda(r_1 - x_i'\beta)}{\partial x_i}\right]$

$= \frac{1}{n}\sum_{i=1}^{n}\left[\Lambda(r_1 - x_i'\beta)(1-\Lambda(r_1 - x_i'\beta))\underline{(-\beta)}\right]$

$= -\frac{\beta}{n}\sum\left[\Lambda(\cdot)(1-\Lambda(\cdot))\right]$

② $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial P(y_i=1|x_i)}{\partial x_i}\right]$

③ $(y_i=2)$

④ $(y_i=3)$

recall (slide 5)
$$\frac{\partial\Lambda(x)}{dx} = \underline{\Lambda(x)(1-\Lambda(x))}$$

$Var(cX) = c^2\, Var(X)$

$Var\left(\frac{e_i}{J}\right) = \underline{\frac{1}{J^2} Var(\varepsilon_i)}$

Consider the Type 1 Tobit model

$$y_i^* = x_i'\beta + \epsilon_i, \quad \epsilon_i | x_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

$\Leftrightarrow \frac{\varepsilon_i}{\sigma} | x_i \sim N(0,1)$

We observe $y_i = 1(y_i^* > 0)\, y_i^*$.

(a) Derive $E[y_i | x_i, y_i > 0]$, $P(y_i > 0 | x_i)$, and $E[y_i | x_i]$. [Hint: $E[y_i | x_i] = E[y_i | x_i, y_i > 0]\, P(y_i > 0 | x_i)$]

$y_i^* > 0 \rightarrow y_i = y_i^* \rightarrow y_i > 0$

$$\begin{aligned}
E[y_i | x_i, y_i > 0] &= E[y_i^* | x_i, y_i^* > 0] \\
&= E[x_i'\beta + \varepsilon_i | x_i, x_i'\beta + \varepsilon_i > 0] \\
&= E[x_i'\beta | x_i, x_i'\beta + \varepsilon_i > 0] + E[\varepsilon_i | x_i, x_i'\beta + \varepsilon_i > 0] \\
&= x_i'\beta + \sigma E\Big[\underbrace{\tfrac{\varepsilon_i}{\sigma}}_{z} \Big| x_i, \underbrace{\tfrac{\varepsilon_i}{\sigma}}_{z} > \underbrace{-\tfrac{x_i'\beta}{\sigma}}_{a}\Big] \\
&= x_i'\beta + \sigma\lambda\left(\tfrac{x_i'\beta}{\sigma}\right) \\
&= x_i'\beta + \sigma\lambda \frac{\phi\left(\frac{x_i'\beta}{\sigma}\right)}{\Phi\left(\frac{x_i'\beta}{\sigma}\right)}
\end{aligned}$$

recall (slide 11)

$z \sim N(0,1)$

a constant

$E[z | z > a] = \frac{\phi(-a)}{\Phi(-a)} = \lambda(-a)$

$$\begin{aligned}
P(y_i > 0 | x_i) = P(y_i^* > 0 | x_i) &= P(\varepsilon_i > -x_i'\beta | x_i) \\
&= P\left(\tfrac{\varepsilon_i}{\sigma} > -\tfrac{x_i'\beta}{\sigma} \Big| x_i\right) \\
&= \Phi\left(\tfrac{x_i'\beta}{\sigma}\right)
\end{aligned}$$