2. Consider the ordered logit model with unobserved utility given by

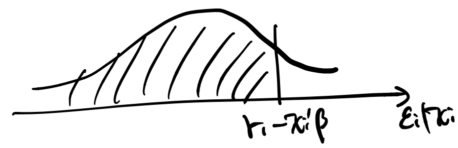
$$y_i^* = x_i'\beta + \epsilon_i, \quad \epsilon_i|x_i \overset{i.i.d.}{\sim} Logistic(0,1)$$

$\Lambda(\cdot)$: CDF of r.v. $\epsilon_i|x_i$

We observe x_i and y_i, where

$$\begin{cases} y_i = 0 \text{ if } y_i^* < \gamma_1 \\ y_i = 1 \text{ if } \gamma_1 \leq y_i^* < \gamma_2 \\ y_i = 2 \text{ if } \gamma_2 \leq y_i^* < \gamma_3 \\ y_i = 3 \text{ if } y_i^* \geq \gamma_3 \end{cases}$$

Our goal is to estimate β, γ_1, γ_2, and γ_3.



Step 1 : setting up conditional prob.

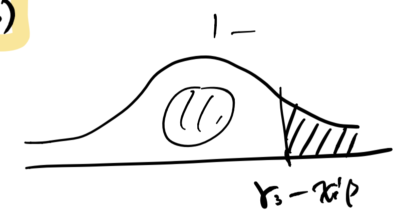
$$P(y_i=0|x_i) = P(y_i^* < \gamma_1|x_i) = P(x_i'\beta + \epsilon_i < \gamma_1|x_i) = P(\epsilon_i < \gamma_1 - x_i'\beta|x_i)$$

$$① = \Lambda(\gamma_1 - x_i'\beta)$$

$$P(y_i=1|x_i) = P(\gamma_1 \leq y_i^* < \gamma_2|x_i) = ② \; \Lambda(\gamma_2 - x_i'\beta) - \Lambda(\gamma_2 - x_i'\beta)$$

$$P(y_i=2|x_i) = ③ \; \Lambda(\gamma_3 - x_i'\beta) - \Lambda(\gamma_2 - x_i'\beta)$$

1 –



$$P(y_i=3|x_i) = P(y_i^* \geq \gamma_3|x_i)$$
$$= P(x_i'\beta + \epsilon_i \geq \gamma_3|x_i)$$
$$= P(\epsilon_i \geq \gamma_3 - x_i'\beta|x_i)$$
$$= 1 - P(\epsilon_i < \gamma_3 - x_i'\beta|x_i) = ④ \; 1 - \Lambda(\gamma_3 - x_i'\beta)$$

Step 2 : Likelihood fn.

$$1(y_i=0) = \begin{cases} 1 & \text{if } y_i = 0 \\ 0 & \text{if } y_i \neq 0 \end{cases}$$

$$L(\gamma_1,\gamma_2,\gamma_3,\beta) = \prod_{i=0}^{n} P(y_i=0|x_i)^{1(y_i=0)} P(y_i=1|x_i)^{1(y_i=1)} P(y_i=2|x_i)^{1(y_i=2)} P(y_i=3|x_i)^{1(y_i=3)}$$

$$= \prod_{y_i=0} \underbrace{P(y_i=0|x_i)}_{①} \prod_{y_i=1} \underbrace{P(y_i=1|x_i)}_{②} \prod_{y_i=2} \underbrace{P(y_i=2|x_i)}_{③} \prod_{y_i=3} \underbrace{P(y_i=3|x_i)}_{④}$$

$$= \prod_{y_i=0} \Lambda(\gamma_1 - x_i'\beta) \prod_{y_i=1} [\; ② \;] \prod_{y_i=2} [\; ③ \;] \prod_{y_i=3} [\; ④ \;]$$

(b) A.M.E.

$$\frac{1}{n}\sum_{i=1}^{n} \boxed{\frac{\partial P(y_i=0|x_i)}{\partial x}}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial \Lambda(\gamma_1 - x_i'\beta)}{\partial x}\right]$$

$$= \left[\frac{\partial \Lambda(\gamma_1 - x_i'\beta)}{\partial(\gamma_1 - x_i'\beta)} \cdot \frac{\partial(\gamma_1 - x_i'\beta)}{\partial x_i}\right]$$

recall $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

recall $\dfrac{\partial \Lambda(x)}{\partial x}$

$$= \quad [\Lambda(\gamma_1 - x_i'\beta)(1-\Lambda(\gamma_1 - x_i'\beta))(-\beta)] \quad (= \Lambda(x)(1-\Lambda(x)))$$

$$= -\frac{\beta}{n}\sum_{i=1}^{n}[\Lambda(\cdot)(1-\Lambda(\cdot))]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial P(y_i=1|x_i)}{\partial x}$$

$$\left(\begin{array}{l} Var(\varepsilon_i) = \sigma^2 \\ Var(\frac{\varepsilon_i}{\sigma}) = 1 \end{array} \right.$$

$$\left(\begin{array}{l} Var(x) = \sigma^2 \\ Var(cx) = c^2\sigma^2 \end{array} \right.$$

Consider the Type 1 Tobit model

$$y_i^* = x_i'\beta + \epsilon_i \qquad \epsilon_i | x_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

$\frac{\varepsilon_i}{\sigma} | x_i \sim N(0,1)$

We observe $y_i = 1(y_i^* > 0)\, y_i^*$.

(a) Derive $E[y_i | x_i, y_i > 0]$, $P(y_i > 0 | x_i)$, and $E[y_i | x_i]$. [Hint: $E[y_i | x_i] = E[y_i | x_i, y_i > 0]\, P(y_i > 0 | x_i)$]

$y_i^* > 0 \rightarrow y_i = y_i^* \rightarrow y_i > 0$

$E[y_i | x_i, y_i > 0]$

$= E[y_i^* | x_i, y_i^* > 0]$

$= E[x_i'\beta + \varepsilon_i | x_i, x_i'\beta + \varepsilon_i > 0]$

$= E[x_i'\beta | x_i, x_i'\beta + \varepsilon_i > 0] + E[\varepsilon_i | x_i, x_i'\beta + \varepsilon_i > 0]$

$= x_i'\beta + E[\varepsilon_i | x_i, x_i'\beta + \varepsilon_i > 0]$

$= \sigma E[\frac{\varepsilon_i}{\sigma} | x_i, \frac{\varepsilon_i}{\sigma} > -\frac{x_i'\beta}{\sigma}]$

z z a

recall (slide #11)

$z \sim N(0,1)$, a constant

$E(z | z > a) = \frac{\phi(-a)}{\Phi(-a)} = \lambda(-a)$

$$= x_i'\beta + \sigma \left[\frac{\phi\left(\frac{x_i'\beta}{\sigma}\right)}{\Phi\left(\frac{x_i'\beta}{\sigma}\right)} \right]$$

$P(y_i > 0 | x_i) = P(y_i^* > 0 | x_i) = P(x_i'\beta + \varepsilon_i > 0 | x_i)$

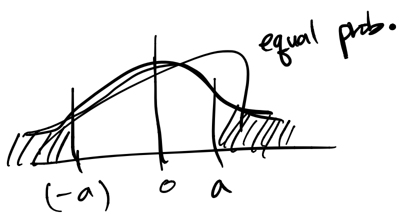
$= P(\varepsilon_i > -x_i'\beta | x_i)$

$= P\left(\frac{\varepsilon_i}{\sigma} > -\frac{x_i'\beta}{\sigma} \middle| x_i\right)$

} because symmetry of dist. & mean zero

$= P\left(\frac{\varepsilon_i}{\sigma} < \frac{x_i'\beta}{\sigma} \middle| x_i\right)$

$= \Phi\left(\frac{x_i'\beta}{\sigma}\right)$



$E[y_i | x_i] = E[y_i | x_i, y_i > 0]\; P(y_i > 0 | x_i)$

$$= \left[x_i'\beta + \sigma \frac{\phi\left(\frac{x_i'\beta}{\sigma}\right)}{\Phi(\cdot)} \right] \Phi(\cdot)$$

$$= x_i'\beta\, \Phi(\cdot) + \sigma\, \phi(\cdot)$$