ECON 211C: Problem Set 4

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Question 1

Deduce the state-space representation for an $AR(p)$ model in [13.1.14] and [13.1.15] and the state-space representation for an $MA(1)$ model given in [13.1.17] and [13.1.18] as special cases of that for the $ARMA(r, r-1)$ model of [13.1.22] and [13.1.23].

Any $ARMA(p, q)$ process can be written in the state-space representation form of $ARMA(r, r-1)$ by defining $r \equiv \max\{p, q+1\}$. The state equation and the observation equation are

State equation ($r = \max\{p, q+1\}$):

$$\boldsymbol{\xi}_{t+1} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{r-1} & \phi_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \boldsymbol{\xi}_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad [13.1.22]$$

Observation equation ($n = 1$):

$$y_t = \mu + \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_{r-1} \end{bmatrix} \boldsymbol{\xi}_t, \qquad [13.1.23]$$

where

$$\boldsymbol{\xi}_t = \begin{bmatrix} (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_t \\ (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_{t-1} \\ \vdots \\ (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_{t-r+1} \end{bmatrix}.$$

For an $AR(p)$ process, we must have

$$\begin{aligned} & y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \varepsilon_t \\ \iff & (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)(y_t - \mu) = \varepsilon_t \\ \iff & y_t - \mu = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1} \varepsilon_t. \end{aligned}$$

Then we have

$$\boldsymbol{\xi}_t = \begin{bmatrix} (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_t \\ (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_{t-1} \\ \vdots \\ (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_{t-r+1} \end{bmatrix} = \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t-r+1} - \mu \end{bmatrix}$$

Therefore, the state-space representation of an $AR(p)$ is

State equation $(r = p)$**:**

$$\begin{bmatrix} y_{t+1}-\mu \\ y_t-\mu \\ \vdots \\ y_{t-p+2}-\mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t-\mu \\ y_{t-1}-\mu \\ \vdots \\ y_{t-p+1}-\mu \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad [13.1.14]$$

Observation equation $(n = 1)$**:**

$$y_t = \mu + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_t-\mu \\ y_{t-1}-\mu \\ \vdots \\ y_{t-p+1}-\mu \end{bmatrix}. \quad [13.1.15]$$

Now, for an $MA(1)$ model, $r = 1 + 1 = 2$ and $\phi_p = 0$ for any p. Then, we have

$$\boldsymbol{\xi}_t = \begin{bmatrix} (1-\phi_1 L - \phi_2 L^2)^{-1}\varepsilon_t \\ (1-\phi_1 L - \phi_2 L^2)^{-1}\varepsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix}$$

Therefore, the state-space representation of an $MA(1)$ is

State equation $(r = 2)$ **:**

$$\begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \quad [13.1.17]$$

Observation equation $(n = 1)$**:**

$$y_t = \mu + \begin{bmatrix} 1 & \theta_1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix}. \quad [13.1.18]$$

Question 2

Derive equation [13.4.5] as a special case of [13.4.1] and [13.4.2] for the model specified in [13.4.3] and [13.4.4] by analysis of the Kalman filter recursion for this case.

From the given model

State equation $(r = 2)$ **:**

$$\boldsymbol{\xi}_{t+1} = \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix} \qquad [13.4.3]$$

Observation equation $(n = 1)$**:**

$$y_t = \varepsilon_{1,t} + \varepsilon_{2,t}, \qquad [13.4.4]$$

we have

$$\boldsymbol{F} = \boldsymbol{0}, \quad \boldsymbol{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}, \quad \boldsymbol{A}' = 0, \quad \boldsymbol{H}' = [1\ 1], \quad \boldsymbol{R} = 0.$$

By Kalman filter recursion,

for $t = 0$

$$\hat{\boldsymbol{\xi}}_{1|0} = E[\boldsymbol{\xi_1}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{P}_{1|0} = E[\xi_1 \xi_1'] = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix},$$

for $t \geq 1$

$$\hat{\boldsymbol{\xi}}_{t+1|t} = \boldsymbol{F}\hat{\boldsymbol{\xi}}_{t|t-1} + \boldsymbol{K_t}(\boldsymbol{y_t} - \boldsymbol{A'x_t} - \boldsymbol{H'}\hat{\boldsymbol{\xi}}_{\boldsymbol{t|t-1}})$$
$$\boldsymbol{P}_{t+1|t} = \boldsymbol{F}\boldsymbol{P_{t|t}}\boldsymbol{F'} + \boldsymbol{Q},$$

where the gain matrix $\boldsymbol{K_t}$ is

$$\boldsymbol{K_t} \equiv \boldsymbol{F}\boldsymbol{P_{t|t}}\boldsymbol{H}(\boldsymbol{H'}\boldsymbol{P_{t|t-1}}\boldsymbol{H} + \boldsymbol{R})^{-\boldsymbol{1}}$$
$$= \boldsymbol{0}.$$

Hence,

$$\hat{\boldsymbol{\xi}}_{t+1|t} = \boldsymbol{0}$$

$$\boldsymbol{P}_{t+1|t} = \boldsymbol{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}.$$

We can substitute values into equation [13.4.1]:

$$\begin{aligned} f_{\boldsymbol{Y}_t|\boldsymbol{X}_T,\boldsymbol{\mathcal{Y}}_{t-1}}(\boldsymbol{y_t}|\boldsymbol{x_t},\boldsymbol{\mathcal{Y}_{t-1}}) &= (2\pi)^{-n/2}|\boldsymbol{H'P_{t|t-1}H+R}|^{-1/2} \\ &\quad\times \exp\left\{-\frac{1}{2}(\boldsymbol{y_t}-\boldsymbol{A'x_t}-\boldsymbol{H'\hat{\xi}_{t|t-1}})'(\boldsymbol{H'P_{t|t-1}H+R})^{-1}(\boldsymbol{y_t}-\boldsymbol{A'x_t}-\boldsymbol{H'\hat{\xi}_{t|t-1}})\right\}. \\ &= (2\pi)^{-1/2}(\sigma_1^2+\sigma_2^2)^{-1/2}\exp\left\{-\frac{1}{2}y_t^2(\sigma_1^2+\sigma_2^2)^{-1}\right\}. \end{aligned}$$

Substituting into the sample log likelihood [13.4.2], we have

$$\begin{aligned} \sum_{t=1}^{T}\log f_{\boldsymbol{Y}_t|\boldsymbol{X}_t,\boldsymbol{\mathcal{Y}}_{t-1}}(\boldsymbol{y_t}|\boldsymbol{x_t},\boldsymbol{\mathcal{Y}_{t-1}}) &= \sum_{t=1}^{T}\log\left\{(2\pi)^{-1/2}(\sigma_1^2+\sigma_2^2)^{-1/2}\exp\left\{-\frac{1}{2}y_t^2(\sigma_1^2+\sigma_2^2)^{-1}\right\}\right\} \\ &= \log\left\{\left((2\pi)^{-1/2}(\sigma_1^2+\sigma_2^2)^{-1/2}\right)^T\right\} - \sum_{t=1}^{T} y_t/[2(\sigma_1^2+\sigma_2^2)] \\ &= -(T/2)\log(2\pi)-(T/2)\log(\sigma_1^2+\sigma_2^2)-\sum_{t=1}^{T} y_t/[2(\sigma_1^2+\sigma_2^2)]. \qquad [13.4.5] \end{aligned}$$