Problem Set 2

Econ 211C

Question 1 .... *35 points*

Suppose $\{Y_i\}_{i=1}^n \overset{i.i.d.}{\sim} Weibull(k, \lambda)$:

$$f_{Y_i}(y_i|k,\lambda) = \frac{k}{\lambda}\left(\frac{y_i}{\lambda}\right)^{k-1} \exp\left\{-\left(\frac{y_i}{\lambda}\right)^k\right\}.$$

(a) (10 points) Derive the log likelihood, $\ell(k, \lambda|\boldsymbol{y})$.

Solution: The likelihood function is equivalent to the joint density:

$$\begin{aligned}
\mathcal{L}(k,\lambda|\boldsymbol{y}) &= f_{\boldsymbol{Y}}(\boldsymbol{y}|k,\lambda) \\
&= \prod_{i=1}^n f_{Y_i}(y_i|k,\lambda) \\
&= \prod_{i=1}^n \frac{k}{\lambda}\left(\frac{y_i}{\lambda}\right)^{k-1} \exp\left\{-\left(\frac{y_i}{\lambda}\right)^k\right\} \\
&= k^n \lambda^{-nk} \exp\left\{-\lambda^{-k}\sum_{i=1}^n y_i^k\right\} \prod_{i=1}^n y_i^{k-1}.
\end{aligned}$$

Thus the log likelihood is

$$\ell(k,\lambda|\boldsymbol{y}) = n\log(k) - nk\log(\lambda) - \lambda^{-k}\sum_{i=1}^n y_i^k + (k-1)\sum_{i=1}^n \log(y_i).$$

(b) (10 points) Derive the maximum likelihood estimators, $\hat{k}$ and $\hat{\lambda}$.

Solution: We begin by taking derivatives of the log likelihood function:

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} - n\log(\lambda) + \log(\lambda)\lambda^{-k}\sum_{i=1}^n y_i^k - \lambda^{-k}\sum_{i=1}^n \log(y_i)y_i^k + \sum_{i=1}^n \log(y_i) \tag{1}$$

$$\frac{\partial \ell}{\partial \lambda} = -\frac{nk}{\lambda} + k\lambda^{-(k+1)}\sum_{i=1}^n y_i^k. \tag{2}$$

Beginning with Equation (2), the MLEs, $\hat{k}$ and $\hat{\lambda}$, are the values such that

$$\begin{aligned}
\left.\frac{\partial \ell}{\partial \lambda}\right|_{\hat{k},\hat{\lambda}} &= 0 \\
\implies n\hat{k}\hat{\lambda}^{-1} &= \hat{k}\hat{\lambda}^{-(\hat{k}+1)}\sum_{i=1}^{n} y_i^{\hat{k}} \\
\implies n\hat{\lambda}^{\hat{k}} &= \sum_{i=1}^{n} y_i^{\hat{k}} \\
\implies \hat{\lambda} &= \left(\frac{1}{n}\sum_{i=1}^{n} y_i^{\hat{k}}\right)^{\frac{1}{\hat{k}}}. \qquad (3)
\end{aligned}$$

Substituting Equation (3) into Equation (1),

$$\left.\frac{\partial \ell}{\partial k}\right|_{\hat{k},\hat{\lambda}} = \frac{n}{\hat{k}} - \frac{n\sum_{i=1}^{n}\log(y_i)y_i^{\hat{k}}}{\sum_{i=1}^{n} y_i^{\hat{k}}} + \sum_{i=1}^{n}\log(y_i) = 0. \qquad (4)$$

Equation (4) defines a unique value for $\hat{k}$, but it must be determined numerically since it can't be solved analytically.

(c) (15 points) Derive the information matrix. What is the observed information matrix? Given estimates, $\hat{k}$ and $\hat{\lambda}$, what would approximations of the variances of the estimates be?

Solution: Beginning with Equations (1) and (2), we derive the second derivatives of the log likelihood function:

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial k^2} &= -\frac{n}{k^2} - \log(\lambda)^2\lambda^{-k}\sum_{i=1}^{n} y_i^k + 2\log(\lambda)\lambda^{-k}\sum_{i=1}^{n}\log(y_i)y_i^k - \lambda^{-k}\sum_{i=1}^{n}\log(y_i)^2 y_i^k \\
\frac{\partial^2 \ell}{\partial k \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial k} &= -\frac{n}{k} + \lambda^{-(k+1)}\left[(1 - k\log(\lambda))\sum_{i=1}^{n} y_i^k + k\sum_{i=1}^{n}\log(y_i)y_i^k\right] \\
\frac{\partial^2 \ell}{\partial \lambda^2} &= nk\lambda^{-2} - k(k+1)\lambda^{-(k+2)}\sum_{i=1}^{n} y_i^k.
\end{aligned}$$

The Hessian is the matrix

$$\mathcal{H}(k,\lambda) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial k^2} & \frac{\partial^2 \ell}{\partial k \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial k} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix},$$

and the information matrix is $\mathcal{I}(k,\lambda) = -\mathrm{E}\left[\mathcal{H}(k,\lambda)^{-1}\right]$. The observed information matrix is simply $-\mathcal{H}(k,\lambda)^{-1}$ and approximations of the variances of $\hat{k}$ and $\hat{\lambda}$ are given by the diagonal elements of $-\mathcal{H}(\hat{k},\hat{\lambda})^{-1}$, where we evaluate the inverse hessian at the estimated values, $\hat{k}$ and $\hat{\lambda}$.

Question 2 .... *35 points*

Consider an $AR(2)$ process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0,1)$ and where $\boldsymbol{\phi} = (\phi_1, \phi_2)' = (1.3, -0.41)'$.

(a) (25 points) Simulate 30 observations from this process and compute the least-squares estimates for three regressions:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \varepsilon_t \\ Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \\ Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + \varepsilon_t. \end{aligned}$$

Repeat the simulation/estimation 1000 times and report the means and standard deviations of each set of estimates in a table. Include your `R` code with your solution.

Solution: The code to generate the results for parts (a), (b) and (c) is shown below:

```
# Parameters
N = 1000
nSim = 100000
phi1 = 1.3
phi2 = -0.41

reg1 = NULL
reg2 = NULL
reg3 = NULL
for(ix in 1:nSim){
```

```
# Simulate
eps = rnorm(N+2)
Y = rep(0,N+2)
for(jx in 3:(N+2)){
Y[jx] = phi1*Y[jx-1]+phi2*Y[jx-2] + eps[jx]
}
Y = Y[3:(N+2)]

# Estimate
reg1 = rbind(reg1,lm(Y[2:N]~Y[1:(N-1)]-1)$coef)
reg2 = rbind(reg2,lm(Y[3:N]~Y[2:(N-1)]+Y[1:(N-2)]-1)$coef)
reg3 = rbind(reg3,lm(Y[4:N]~Y[3:(N-1)]+Y[2:(N-2)]+Y[1:(N-3)]-1)$coef)
}

# Compute moments
signif(apply(reg1,2,mean),4)
signif(apply(reg1,2,sd),4)
signif(apply(reg2,2,mean),4)
signif(apply(reg2,2,sd),4)
signif(apply(reg3,2,mean),4)
signif(apply(reg3,2,sd),4)
```

The only modification that must be made for each part is on the second line, modifying the value of N. The corresponding results are reported in the table below.

N	ϕ_1	ϕ_2	ϕ_3
		$AR(1)$	
30	0.9089 (0.06614)	NA	NA
1000	0.9215 (0.008059)	NA	NA
		$AR(2)$	
30	1.265 (0.1826)	-0.4090 (0.1789)	NA
1000	1.299 (0.02910)	-0.4105 (0.02889)	NA
		$AR(3)$	
30	1.264 (0.2007)	-0.4110 (0.2918)	0.0004006 (0.1898)
1000	1.299 (0.03196)	-0.4105 (0.05125)	0.00001385 (0.03263)

(b) (5 points) Repeat part (a), simulating $N = 1000$ observations instead of $N = 30$ observations at each iteration. Report the estimates in the same table as part (a).

(c) (5 points) Repeat part (b) with $N = 100,000$.

Question 3 .... *30 points*

Download daily adjusted closing prices for ticker XIV from finance.yahoo.com for dates 25 April 2014 to 24 April 2015. Find the best fitting $ARMA$ model for these data. Report the parameter estimates and standard errors and provide some interpretation.

Solution: The code to load data and fit $ARMA$ models is shown below.

```
library(quantmod)
getSymbols('XIV',from='2014-04-25',to='2015-04-24')
rets = dailyReturn(XIV)
auto.arima(rets)
```

In this case, I used the `auto.arima` fuction from the `forecast` package in `R` to sequentially search over $ARMA(p, q)$ models for $p = 0, \ldots, 5$ and $q = 0, \ldots, 5$. The function can use a variety of information criteria (penalized log likelihoods) to select the best fitting model, which balances the objectives of maximizing the likelihood and maintaining a low order of parameterization. Using BIC (Bayeisan Information Criterion), `auto.arima` selected an $ARMA(0, 0)$ with zero mean as the best fitting model. This suggests that daily XIV returns are white noise, or that daily XIV prices are a random walk.