ECON 211C: Problem Set 1

Due: Tuesday, April 25, 2017

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Question 1

(a)

The autocovariance of the $MA(2)$ process $Y_t = (1 + 2.4L + 0.8L^2)\varepsilon_t$ is

$$\begin{aligned}\gamma_{jt} &= E[(Y_t - \mu)(Y_{t-j} - \mu)] \\ &= E[Y_t Y_{t-j}] \\ &= E[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_{t-j} + 2.4\varepsilon_{t-j-1} + 0.8\varepsilon_{t-j-2})].\end{aligned}$$

The autocovariances can be computed as

$$\begin{aligned}\gamma_0 &= E[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})] \\ &= E[\varepsilon_t^2 + 2.4^2\varepsilon_{t-1}^2 + 0.8^2\varepsilon_{t-2}^2] \\ &= 1 + 5.76 + 0.64 = 7.4 \\ \gamma_1 &= E[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_{t-1} + 2.4\varepsilon_{t-2} + 0.8\varepsilon_{t-3})] \\ &= E[2.4\varepsilon_{t-1}^2 + 0.8 \cdot 2.4\varepsilon_{t-2}^2] \\ &= 2.4 + 1.92 = 4.32 \\ \gamma_2 &= E[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_{t-2} + 2.4\varepsilon_{t-3} + 0.8\varepsilon_{t-4})] \\ &= E[0.8\varepsilon_{t-2}^2] \\ &= 0.8 \\ \gamma_3 &= E[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_{t-3} + 2.4\varepsilon_{t-4} + 0.8\varepsilon_{t-5})] \\ &= 0 \\ \gamma_4 &= 0 \\ &\vdots\end{aligned}$$

Therefore, we have

$$\gamma_0 = 7.4, \quad \gamma_1 = 4.32, \quad \gamma_2 = 0.8, \quad \text{and} \quad \gamma_j = 0 \quad \forall j \geq 3$$

Since the autocovariances do not depend on t, Y_t is weakly stationary, or covariance stationary.

(b)

The MA process is invertible, if all the roots of the lag polynomial lie outside the unit circle (i.e. all the eigenvalues lie inside the unit circle). Thus, for being not invertible we want to show that at least one of the roots lie inside the unit circle. By the quadratic formula, the roots are

$$\lambda_1 = \frac{-2.4 + \sqrt{2.4^2 - 4 \cdot 0.8}}{1.6} = -0.5 \quad \text{and} \quad \lambda_2 = \frac{-2.4 - \sqrt{2.4^2 - 4 \cdot 0.8}}{1.6} = -2.5,$$

and the polynomial can be factored as

$$\theta(L) = (1 + 2L)(1 + 0.4L).$$

Since $|\lambda_1| < 1$, Y_t is not invertible.
Yet, as none of the roots lie *on* the unit circle, Y_t can be uniquely represented as an invertible form

$$\begin{aligned}\widetilde{Y}_t &= \widetilde{\theta}(L)\widetilde{\varepsilon}_t \\ &= (1+0.5L)(1+0.4)\widetilde{\varepsilon}_t \\ &= (1+0.9L+0.2L^2)\widetilde{\varepsilon}_t,\end{aligned}$$

where $\widetilde{\varepsilon}_t \sim WN(0, 2^2)$.

(c)

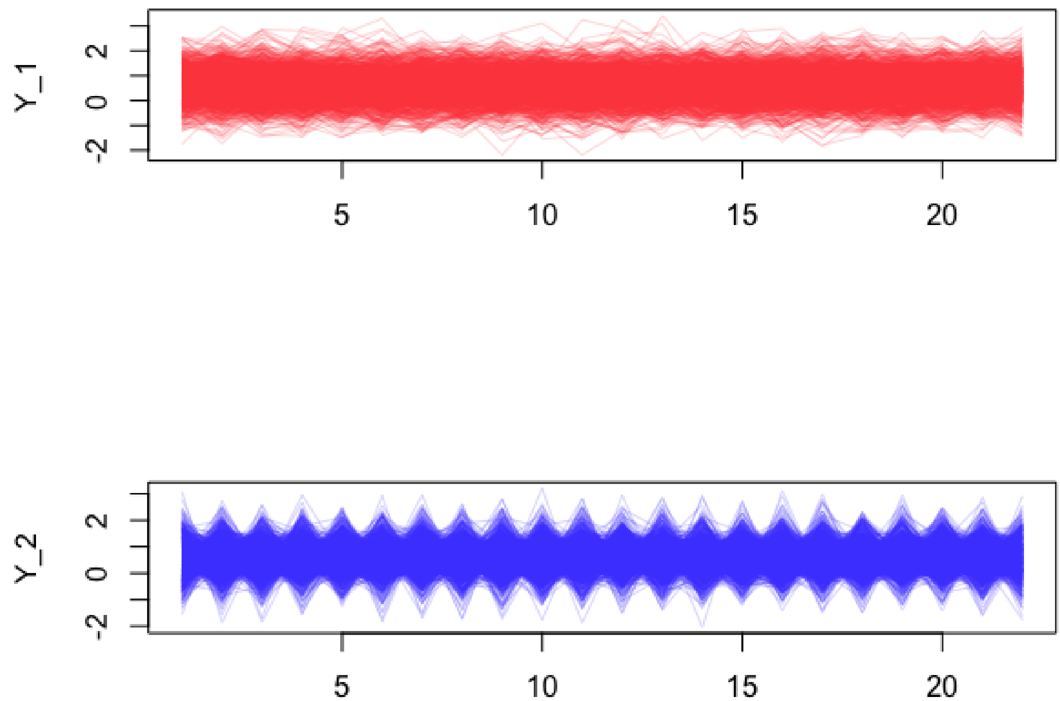
Given that $E[\widetilde{\varepsilon}_t^2] = 4$, the autocovariances of $\widetilde{Y}_t$ can be computed as

$$\begin{aligned}\widetilde{\gamma}_0 &= E[\varepsilon_t^2 + 0.9^2\varepsilon_{t-1}^2 + 0.2^2\varepsilon_{t-2}^2] = 7.4 \\ \widetilde{\gamma}_1 &= E[0.9\varepsilon_{t-1}^2 + 0.2\cdot 0.9\varepsilon_{t-2}^2] = 4.32 \\ \widetilde{\gamma}_2 &= E[0.2\varepsilon_{t-2}^2] = 0.8 \\ \widetilde{\gamma}_j &= 0 \quad \forall j \geq 3,\end{aligned}$$

which are equivalent to those in part (a).

Question 2

Figure 1: Comparision of two MA(1) processes



See Appendix for the R code.

Question 3

(a)

The $ARMA(2,5)$ process can be expressed in lag operators:

$$Y_t = 1.3Y_{t-1} - 0.4Y_{t-2} + \varepsilon_t + 0.7\varepsilon_{t-1} + 0.1\varepsilon_{t-3} - 0.5\varepsilon_{t-4} - 0.2\varepsilon_{t-5}$$

$$\Leftrightarrow Y_t - 1.3Y_{t-1} + 0.4Y_{t-2} = \varepsilon_t + 0.7\varepsilon_{t-1} + 0.1\varepsilon_{t-3} - 0.5\varepsilon_{t-4} - 0.2\varepsilon_{t-5}$$
$$\Leftrightarrow (1 - 1.3L + 0.4L^2)Y_t = (1 + 0.7L + 0.1L^3 - 0.5L^4 - 0.2L^5)\varepsilon_t$$
$$\Leftrightarrow \phi(L)Y_t = \theta(L)\varepsilon_t.$$

The $ARMA$ process is weakly stationary if the roots of $\phi(L)$ all lie outside the unit circle. By the quadratic formula, the roots are

$$\lambda_1 = \frac{-1.3 + \sqrt{1.3^2 - 4 \cdot 0.4}}{0.8} = -1.25 \quad \text{and} \quad \lambda_2 = \frac{-1.3 - \sqrt{1.3^2 - 4 \cdot 0.4}}{0.8} = -2.$$

Since both roots lie outside the unit circle, the process is stationary.

(b)

The $ARMA$ process is invertible if all of the roots of $\theta(L)$ lie outside the unit circle. Our process can be rewritten as

$$\boldsymbol{\varepsilon}_t = \Theta\boldsymbol{\varepsilon}_{t-1} + \mathbf{Y}_t,$$

where

$$\boldsymbol{\varepsilon}_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \varepsilon_{t-3} \\ \varepsilon_{t-4} \end{bmatrix} \quad \Theta = \begin{bmatrix} -0.7 & 0 & -0.1 & 0.5 & 0.2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{Y}_t = \begin{bmatrix} Y_t - 1.3Y_{t-1} + 0.4Y_{t-2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalue of Θ are the inverses of the roots of $\theta(L)$. Therefore, we need all the eigenvalues of Θ to lie inside the unit circle. The eigenvalues are computed to be approximately -1.05, -0.47, -0.41, -0.02, -0.66. Since one of them lie outside of the unit circle, the process is not invertible.

(c)

$$\gamma_j = E[Y_tY_{t-j}]$$

$$Y_t = 1.3Y_{t-1} - 0.4Y_{t-2} + \varepsilon_t + 0.7\varepsilon_{t-1} + 0.1\varepsilon_{t-3} - 0.5\varepsilon_{t-4} - 0.2\varepsilon_{t-5}$$

The first five autocovariances for Y_t can be computed as

$$\begin{aligned}
E[Y_tY_t] &= 1.3E[Y_tY_{t-1}] - 0.4E[Y_tY_{t-2}] + E[Y_t\varepsilon_t] + 0.7E[Y_t\varepsilon_{t-1}] + 0.1E[Y_t\varepsilon_{t-3}] - 0.5E[Y_t\varepsilon_{t-4}] - 0.2E[Y_t\varepsilon_{t-5}] \\
&= 1.3\gamma_1 - 0.4\gamma_2 + (1 + 1.4 + 0.216 - 0.714 - 0.15848) = \boxed{1.3\gamma_1 - 0.4\gamma_2 + 1.74352 = \gamma_0} \\
E[Y_tY_{t-1}] &= 1.3E[Y_{t-1}Y_{t-1}] - 0.4E[Y_{t-1}Y_{t-2}] + 0.7E[Y_{t-1}\varepsilon_{t-1}] + 0.1E[Y_{t-1}\varepsilon_{t-3}] - 0.5E[Y_{t-1}\varepsilon_{t-4}] - 0.2E[Y_{t-1}\varepsilon_{t-5}] \\
&= 1.3\gamma_0 - 0.4\gamma_1 + (0.7 + 0.22 - 1.08 - 0.2856) = \boxed{1.3\gamma_0 - 0.4\gamma_1 - 0.4456 = \gamma_1} \\
E[Y_tY_{t-2}] &= 1.3E[Y_{t-2}Y_{t-1}] - 0.4E[Y_{t-2}Y_{t-2}] + 0.1E[Y_{t-2}\varepsilon_{t-3}] - 0.5E[Y_{t-2}\varepsilon_{t-4}] - 0.2E[Y_{t-2}\varepsilon_{t-5}] \\
&= 1.3\gamma_1 - 0.4\gamma_0 + (0.2 - 1.1 - 0.432) = \boxed{1.3\gamma_1 - 0.4\gamma_0 - 1.332 = \gamma_2} \\
E[Y_tY_{t-3}] &= 1.3E[Y_{t-3}Y_{t-1}] - 0.4E[Y_{t-3}Y_{t-2}] + 0.1E[Y_{t-3}\varepsilon_{t-3}] - 0.5E[Y_{t-3}\varepsilon_{t-4}] - 0.2E[Y_{t-3}\varepsilon_{t-5}] \\
&= 1.3\gamma_2 - 0.4\gamma_1 + (0.1 - 1 - 0.44) = \boxed{1.3\gamma_2 - 0.4\gamma_1 - 1.34 = \gamma_3} \\
E[Y_tY_{t-4}] &= 1.3E[Y_{t-4}Y_{t-1}] - 0.4E[Y_{t-4}Y_{t-2}] - 0.5E[Y_{t-4}\varepsilon_{t-4}] - 0.2E[Y_{t-4}\varepsilon_{t-5}] \\
&= 1.3\gamma_3 - 0.4\gamma_2 + (-0.5 - 0.4) = \boxed{1.3\gamma_3 - 0.4\gamma_2 - 0.9 = \gamma_4} \\
E[Y_tY_{t-5}] &= 1.3E[Y_{t-5}Y_{t-1}] - 0.4E[Y_{t-5}Y_{t-2}] - 0.2E[Y_{t-5}\varepsilon_{t-5}] \\
&= \boxed{1.3\gamma_4 - 0.4\gamma_3 - 0.2 = \gamma_5},
\end{aligned}$$

which can be solved as

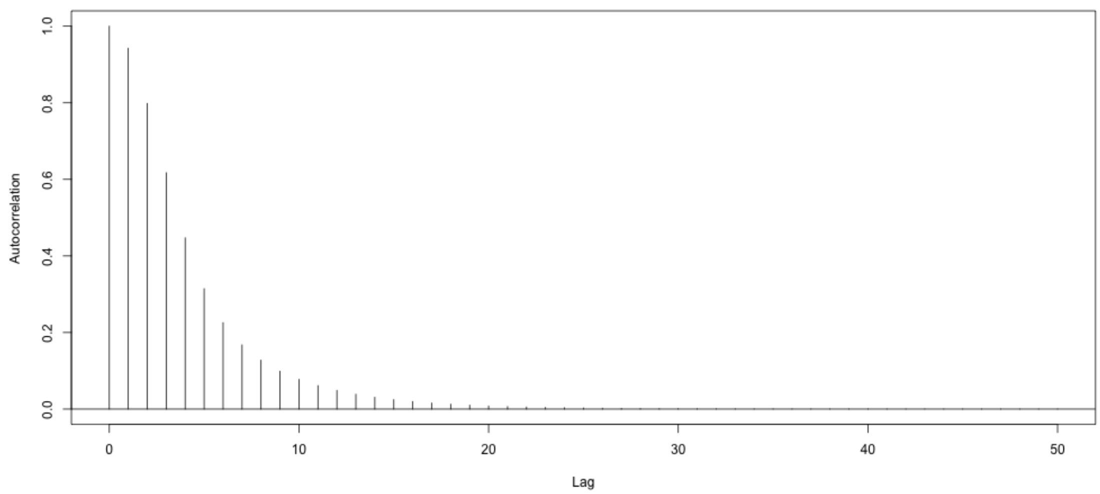
$$\begin{aligned}
\gamma_0 &= 17.517 \\
\gamma_1 &= 15.957 \\
\gamma_2 &= 12.401 \\
\gamma_3 &= 8.396 \\
\gamma_4 &= 5.058 \\
\gamma_5 &= 3.016.
\end{aligned}$$

The autocovariances for $j > 5$ are independent of any values of ε. Therefore, the recursive equation is

$$\gamma_j = 1.3\gamma_{j-1} - 0.4\gamma_{j-2}.$$

(d)

Figure 2: Autocorrelation of $ARMA(2,5)$



(e)

Table 1 shows the sample mean, variance and autocovariances estimated from the simulation results. The first five sample autocovariances are quite close to the theoretical counterparts in part (c) but not enough.

Table 1: Sample mean,variance and autocovariance from simulation (N=1,000)

N	$\hat{\mu}$	$\hat{\sigma^2}$	$\hat{\gamma_0}$	$\hat{\gamma_1}$	$\hat{\gamma_2}$	$\hat{\gamma_3}$	$\hat{\gamma_4}$	$\hat{\gamma_5}$
1,000	0.009	15.514	15.498	13.870	10.222	6.304	3.401	2.133

(f)

Table 2 displays the estimates of mean, variance, and autocovariances from higher number of simulations. As N increases, the estimated autocovariances become closer to the theoretical values.

Table 2: Sample mean, variance and autocovariance from simulation (N=10,000; 100,000; 1,000,000)

N	$\hat{\mu}$	$\hat{\sigma^2}$	$\hat{\gamma_0}$	$\hat{\gamma_1}$	$\hat{\gamma_2}$	$\hat{\gamma_3}$	$\hat{\gamma_4}$	$\hat{\gamma_5}$
10,000	-0.215	17.242	17.240	15.662	12.103	8.123	4.865	2.963
100,000	0.048	17.851	17.851	16.282	12.733	8.742	5.411	3.371
1,000,000	-0.022	17.524	17.524	15.954	12.399	8.400	5.062	3.018

Appendix: R Code

```
######################################## Question 2 #######################################
# Settings
N = 1000
T = 23
mu = 0.61
sigma = 0.5
theta = 0.95
eps = matrix(rnorm(T*N,0,sigma), nrow = T, ncol = N)

# Simulation
Y_1 = mu + eps[2:T,] + theta * eps[1:(T-1),]
Y_2 = mu + eps[2:T,] - theta * eps[1:(T-1),]

# Plot
png(filename="211c_ps1_fig_2.png")
par(mfrow=c(2,1))
matplot(Y_1,type="l",lty=1,col=rgb(1,0.3,0.3,0.2))
matplot(Y_2,type="l",lty=1,col=rgb(0.3,0.3,1,0.2))
dev.off()

##################################### Question 3 (b) #####################################
Theta = matrix(c(-0.7, 0, -0.1, 0.5, 0.2, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0 ,
0, 0, 0, 1, 0),5,5,byrow=TRUE)
EEE = eigen(Theta)
View(EEE)

##################################### Question 3 (d) #####################################
YYY = ARMAacf(ar= c(1.3, -0.4), ma = c(1, 0.7, 0, 0.1, -0.5, -0.2), lag.max = 50)
png(filename="211c_ps1_fig_3d.png",width = 1000, height = 500)
plot(YYY, x = 0:50, ylim=range(c(0,1)), type = "h", xlab = "Lag", ylab = "Autocorrelation")
abline(h = 0)
dev.off()

# Autocorrelations export to TeX file
library(stargazer)
stargazer(YYY)

##################################### Question 3 (e) #####################################
# Settings
phi = c(1.3, -0.4)
theta = c(1,0.7, 0, 0.1, -0.5, -0.2)
```

```
N = 1000 # Number of simulations
eps = rnorm(N+5)
Y = rep(0,N+5)

# Simulation
for(i in 6:length(Y)){
  Y[i] = phi %*% Y[(i-1):(i-2)] + theta %*% eps[i:(i-5)]
  }
Y = Y[-(1:5)]

# Returning sample mean, variance, and autocovariance
mu_e3 = mean(Y)
sig_e3 = var(Y)
rho_e3 =acf(Y,plot=FALSE,lag.max=5,type="covariance")$acf
n_e3 = c(mu_e3,sig_e3,rho_e3)

# Exporting to LaTeX
Q3_e = matrix(c(n_e3),1,8,byrow = TRUE)
xtable(Q3_e, digits = 3)

#################################### Question 3 (f) ####################################
# N=10,000
N = 10000
eps = rnorm(N+5)
Y = rep(0,N+5)
for(i in 6:length(Y)){
  Y[i] = phi %*% Y[(i-1):(i-2)] + theta %*% eps[i:(i-5)]
  }
Y = Y[-(1:5)]
mu_e4 = mean(Y)
sig_e4 = var(Y)
rho_e4 =acf(Y,plot=FALSE,lag.max=5,type="covariance")$acf
n_e4 = c(mu_e4,sig_e4,rho_e4)

# N=100,000
N = 100000
eps = rnorm(N+5)
Y = rep(0,N+5)
for(i in 6:length(Y)){
  Y[i] = phi %*% Y[(i-1):(i-2)] + theta %*% eps[i:(i-5)]
  }
Y = Y[-(1:5)]
mu_e5 = mean(Y)
```

```
sig_e5 = var(Y)
rho_e5 =acf(Y,plot=FALSE,lag.max=5,type="covariance")$acf
n_e5 = c(mu_e5,sig_e5,rho_e5)

# N=1,000,000
N = 1000000
eps = rnorm(N+5)
Y = rep(0,N+5)
for(i in 6:length(Y)){
  Y[i] = phi %*% Y[(i-1):(i-2)] + theta %*% eps[i:(i-5)]
  }
Y = Y[-(1:5)]
mu_e6 = mean(Y)
sig_e6 = var(Y)
rho_e6 =acf(Y,plot=FALSE,lag.max=5,type="covariance")$acf
n_e6 = c(mu_e6,sig_e6,rho_e6)


# Exporting to LaTeX
Q3_f = matrix(c(n_e4,n_e5,n_e6),3,8,byrow = TRUE)
xtable(Q3_f, digits = 3)
```