Problem Set 1

Econ 211C

Question 1 .... *25 points*

[Hamilton, Exercises 3.1 & 3.8] Consider the following $MA(2)$ process:

$$Y_t = (1 + 2.4L + 0.8L^2)\varepsilon_t,$$

where $\varepsilon_t \sim WN(0, 1)$.

(a) (5 points) Is the process weakly stationary? If so, calculate its autocovariances.

Solution: A finite-order MA process is *always* stationary. The mean is

$$\mathrm{E}[Y_t] = \mathrm{E}\left[(1 + 2.4L + 0.8L^2)\varepsilon_t\right] = 0.$$

The autocovariances are defined by

$$\begin{aligned}
\gamma_j &= \mathrm{E}\left[(Y_t - \mu)(Y_{t-j} - \mu)\right] \\
&= \mathrm{E}\left[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_{t-j} + 2.4\varepsilon_{t-1-j} + 0.8\varepsilon_{t-2-j})\right] \\
&= \mathrm{E}[\varepsilon_t\varepsilon_{t-j} + 2.4\varepsilon_t\varepsilon_{t-1-j} + 0.8\varepsilon_{t-2}\varepsilon_{t-2-j} \\
&\quad + 2.4\varepsilon_{t-1}\varepsilon_{t-j} + 5.76\varepsilon_{t-1}\varepsilon_{t-1-j} + 1.92\varepsilon_{t-1}\varepsilon_{t-2-j} \\
&\qquad + 0.8\varepsilon_{t-2}\varepsilon_{t-j} + 1.92\varepsilon_{t-2}\varepsilon_{t-1-j} + 0.64\varepsilon_{t-2}\varepsilon_{t-2-j}]
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma_0 &= \mathrm{E}[\varepsilon_t^2 + 5.76\varepsilon_{t-1}^2 + 0.64\varepsilon_{t-2}^2] = 7.4 \\
\gamma_1 &= \mathrm{E}[2.4\varepsilon_{t-1}^2 + 1.92\varepsilon_{t-2}^2] = 4.32 \\
\gamma_2 &= \mathrm{E}[0.8\varepsilon_{t-2}^2] = 0.8 \\
\gamma_j &= 0, \qquad j = 3, 4, \ldots.
\end{aligned}$$

(b) (10 points) Show that the process is not invertible and find an invertible representation for the process.

Solution: To prove that the process the is invertible, we need to show that the roots of the $MA(2)$ lag polynomial

$$1 + 2.4L + 0.8L^2 = 0$$

lie outside the unit circle. We can factor the polynomial as

$$1 + 2.4L + 0.8L^2 = (1 + 2L)(1 + 0.4L).$$

The resulting roots are

$$\left|\frac{1}{\lambda_1}\right| = \left|-\frac{1}{2}\right| = |-0.5| < 1 \qquad \text{and} \qquad \left|\frac{1}{\lambda_2}\right| = \left|-\frac{1}{0.4}\right| = |-2.5| > 1.$$

Since one of the roots lies inside the unit circle, the process is not invertible. The invertible representation of the $MA(2)$ is

$$\tilde{Y}_t = (1 + 0.4L)(1 + 2^{-1}L)\tilde{\varepsilon}_t = (1 + 0.9L + 0.2L^2)\tilde{\varepsilon}_t,$$

where $\tilde{\varepsilon}_t \sim WN(0, 4)$.

(c) (10 points) Calculate the autocovariances of the invertible representation. How do these relate to the autocovariances in part (a)?

Solution: Substituting $\tilde{\varepsilon}_t$ for ε_t in the solution for part (a), we find Specifically the autocovariances are

$$\begin{aligned}
\tilde{\gamma}_0 &= 4(1 + 0.9^2 + 0.2^2) = 7.4 \\
\tilde{\gamma}_1 &= 4(0.9 + 0.2 \times 0.9) = 4.32 \\
\tilde{\gamma}_2 &= 4(0.2) = 0.8 \\
\tilde{\gamma}_j &= 0, \qquad j = 3, 4, \ldots.
\end{aligned}$$

These autocovarinces are equivalent to the autocovariances in part (a).

Question 2 .... *33 points*

Consider two $MA(1)$ processes that differ only in the sign of their MA coefficient:

$$Y_{1,t} = 0.61 + \varepsilon_t + 0.95\varepsilon_{t-1}$$
$$Y_{2,t} = 0.61 + \varepsilon_t - 0.95\varepsilon_{t-1}$$

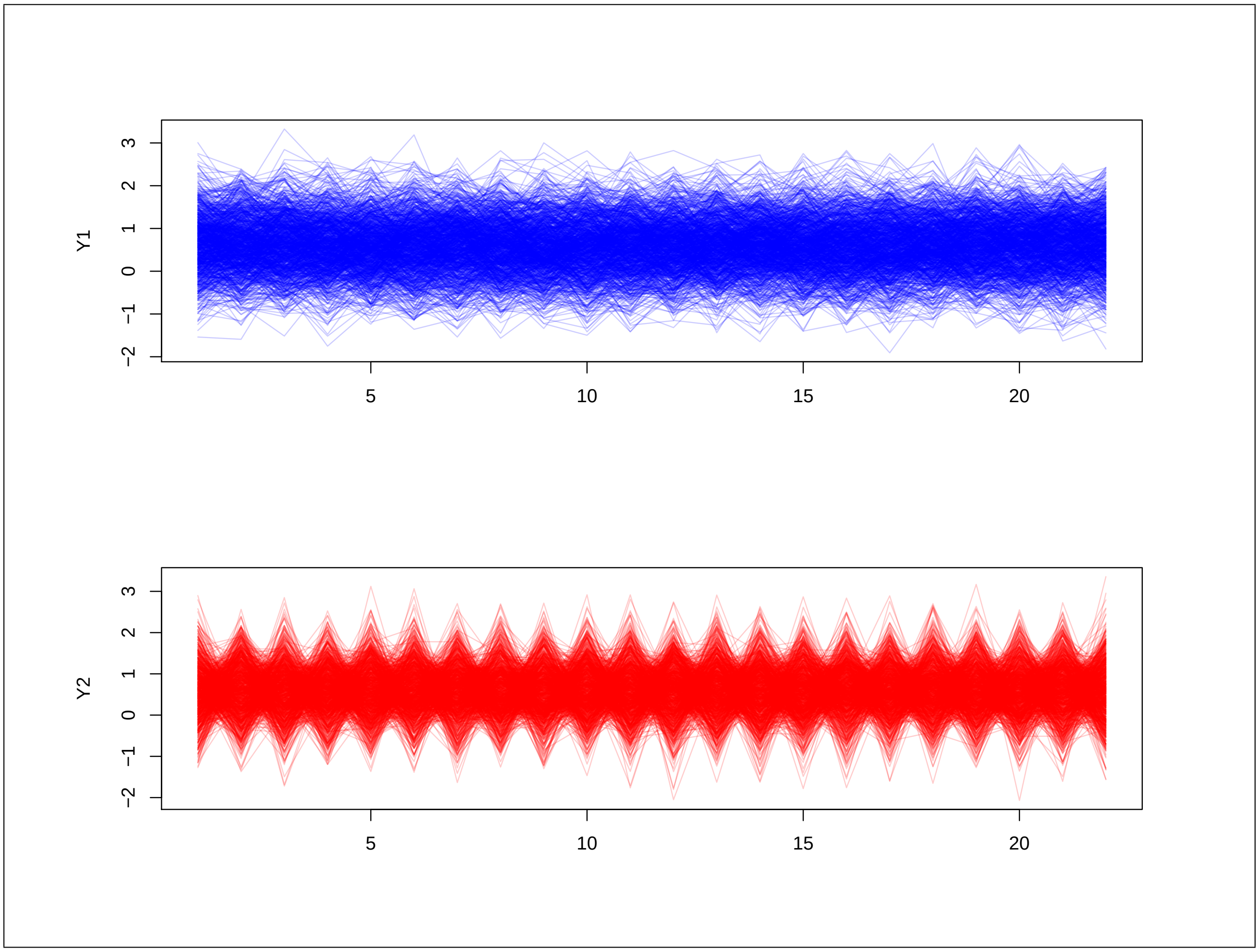
where $\varepsilon_t \sim WN(0, \sigma = 0.5)$. Simulate 1000 instances of 23 observations ($t = 1, 2, \ldots, 23$) of Y_1 and Y_2. For each simulation of $\{\varepsilon\}_{t=0}^{23}$, make sure to use *the same* values to compute Y_1 and Y_2 (i.e. do not simulate different ε sequences for the two MA processes). Plot the groups of time series in two different panels of a single figure: plot all time paths of Y_1 in the upper panel and all time paths of Y_2 in the lower panel. Set the transparency of each line to 0.2. You will find the `rgb` function to be useful in order to pass a color (and transparency value) to the `plot` function. *Bonus (5 points):* Write your code (including plotting) with no loops.

Solution:

```
# Setup
nSim = 1000
N = 23
mu = 0.61
sigma = 0.5
eps = rnorm(N, 0, sigma)
theta1 = 0.95
theta2 = -0.95

# Simulate
eps = matrix(rnorm(N*nSim,0,sigma),nrow=N,ncol=nSim)
Y1 = rep(mu,N-1) + eps[2:N,] + theta1*eps[1:(N-1),]
Y2 = rep(mu,N-1) + eps[2:N,] + theta2*eps[1:(N-1),]

# Plots
par(mfrow=c(2,1))
matplot(1:(N-1),Y1,type='l', lty=1, col=rgb(0, 0, 1, 0.2), xlab="")
matplot(1:(N-1),Y2,type='l', lty=1, col=rgb(1, 0, 0, 0.2), xlab="")
```



Question 3 .... *42 points*

Consider the $ARMA(2,5)$ model,

$$Y_t = 1.3Y_{t-1} - 0.4Y_{t-2} + \varepsilon_t + 0.7\varepsilon_{t-1} + 0.1\varepsilon_{t-3} - 0.5\varepsilon_{t-4} - 0.2\varepsilon_{t-5},$$

where $\varepsilon_t \sim WN(0,1)$.

(a) (7 points) Is this $ARMA$ process for Y_t weakly stationary?

> **Solution:** The $ARMA$ process is stationary if all of the roots of the charactistic polynomial associated with the AR parameters lie outside the unit circle. The relevent polynomial is
>
> $$1 - 1.3z - 0.4z^2 = 0,$$

which has roots

$$z_1 = \frac{1.3 + \sqrt{1.69 - 1.6}}{0.8} = 2$$
$$z_2 = \frac{1.3 - \sqrt{1.69 - 1.6}}{0.8} = \frac{5}{4}.$$

Since both roots lie outside of the unit circle, we conclude that the process is stationary.

(b) (7 points) Is this $ARMA$ process for Y_t invertible?

Solution: Invertibility is guaranteed if all of the roots of the characteristic polynomial associated with the MA parameters lie outside the unit circle. Equivalently, by expressing the process as a vector $AR(1)$ in terms of the lagged innovations, invertibility is guaranteed if the eigenvalues of the coefficient matrix are all within the unit circle (this is the same way we would determine stationarity if the process were expressed as a vector $AR(1)$, in terms of the observed data). Hence, we define

$$\boldsymbol{\xi}_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \varepsilon_{t-3} \\ \varepsilon_{t-4} \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} -0.7 & 0 & -0.1 & 0.5 & 0.2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_t = \begin{bmatrix} w_t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $w_t = Y_t - 1.3Y_{t-1} + 0.4Y_{t-2}$. The $ARMA(2,5)$ process can now be expressed as

$$\boldsymbol{\xi}_t = \boldsymbol{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{v}_t.$$

and invertibility is guaranteed if all of the eigenvalues of $\boldsymbol{F}$ lie within the unit circle. Using `R`, we find that the five eigenvalues of $\boldsymbol{F}$ have moduli 1.0546, 0.8229, 0.8229, 0.7733 and 0.3621. Since the eigenvalue with largest modulus is outside of the unit circle, we conclude that the process is not invertible.

(c) (7 points) Calculate the first 5 autocovariances for Y_t. Derive a recursive equation that can be used to compute all subsequent autocovariances.

Solution: The burden of this exercise falls on computing the first 5 autocovariances; for $j > 4$, the process behaves as an $AR(2)$. We thus begin with the following useful computations:

$$\begin{aligned}
\mathrm{E}\left[Y_t\varepsilon_t\right] &= 1 \\
\mathrm{E}\left[Y_t\varepsilon_{t-1}\right] &= 1.3\mathrm{E}\left[Y_{t-1}\varepsilon_{t-1}\right] + 0.7\mathrm{E}\left[\varepsilon_{t-1}^2\right] \;=\; 2 \\
\mathrm{E}\left[Y_t\varepsilon_{t-2}\right] &= 1.3\mathrm{E}\left[Y_{t-1}\varepsilon_{t-2}\right] - 0.4\mathrm{E}\left[Y_{t-2}\varepsilon_{t-2}\right] \;=\; 2.2 \\
\mathrm{E}\left[Y_t\varepsilon_{t-3}\right] &= 1.3\mathrm{E}\left[Y_{t-1}\varepsilon_{t-3}\right] - 0.4\mathrm{E}\left[Y_{t-2}\varepsilon_{t-3}\right] + 0.1\mathrm{E}\left[\varepsilon_{t-3}^2\right] \;=\; 2.16 \\
\mathrm{E}\left[Y_t\varepsilon_{t-4}\right] &= 1.3\mathrm{E}\left[Y_{t-1}\varepsilon_{t-4}\right] - 0.4\mathrm{E}\left[Y_{t-2}\varepsilon_{t-4}\right] - 0.5\mathrm{E}\left[\varepsilon_{t-4}^2\right] \;=\; 1.428 \\
\mathrm{E}\left[Y_t\varepsilon_{t-5}\right] &= 1.3\mathrm{E}\left[Y_{t-1}\varepsilon_{t-5}\right] - 0.4\mathrm{E}\left[Y_{t-2}\varepsilon_{t-5}\right] - 0.2\mathrm{E}\left[\varepsilon_{t-5}^2\right] \;=\; 0.7924.
\end{aligned}$$

By multiplying the $ARMA(2,5)$ epression with successive lags of Y_t and noting that $\mathrm{E}\left[Y_s\varepsilon_j\right] = 0$ for $j > s$, we find the first five autocovariances to be

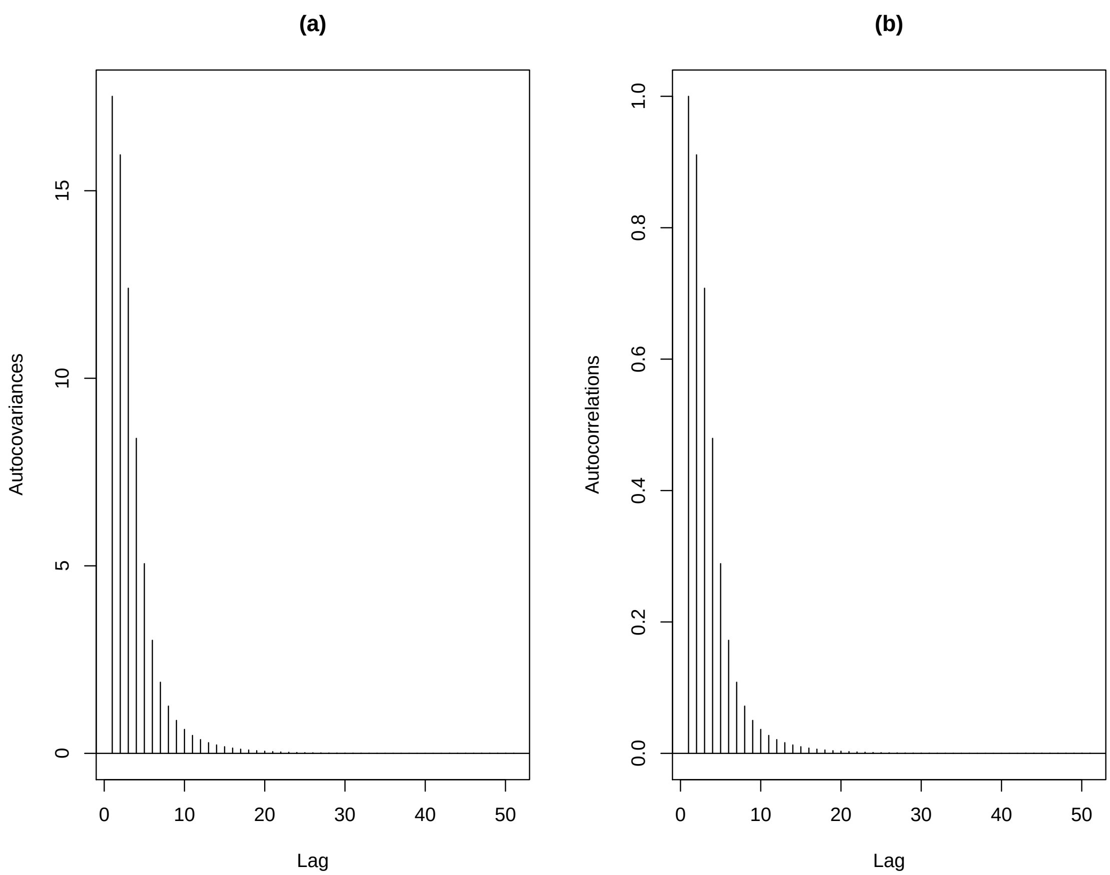
$$\begin{aligned}
\gamma_0 &= 1.3\mathrm{E}\left[Y_tY_{t-1}\right] - 0.4\mathrm{E}\left[Y_tY_{t-2}\right] + \mathrm{E}\left[Y_t\varepsilon_t\right] + 0.7\mathrm{E}\left[Y_t\varepsilon_{t-1}\right] \\
&\qquad + 0.1\mathrm{E}\left[Y_t\varepsilon_{t-3}\right] - 0.5\mathrm{E}\left[Y_t\varepsilon_{t-4}\right] - 0.2\mathrm{E}\left[Y_t\varepsilon_{t-5}\right] \\
&= 1.3\gamma_1 - 0.4\gamma_2 + 1.74352 \qquad (1) \\
\gamma_1 &= 1.3\mathrm{E}\left[Y_{t-1}^2\right] - 0.4\mathrm{E}\left[Y_{t-1}Y_{t-2}\right] + 0.7\mathrm{E}\left[Y_{t-1}\varepsilon_{t-1}\right] \\
&\qquad + 0.1\mathrm{E}\left[Y_{t-1}\varepsilon_{t-3}\right] - 0.5\mathrm{E}\left[Y_{t-1}\varepsilon_{t-4}\right] - 0.2\mathrm{E}\left[Y_{t-1}\varepsilon_{t-5}\right] \\
&= 1.3\gamma_0 - 0.4\gamma_1 - 0.4456 \qquad (2) \\
\gamma_2 &= 1.3\mathrm{E}\left[Y_{t-1}Y_{t-2}\right] - 0.4\mathrm{E}\left[Y_{t-2}Y_{t-2}\right] \\
&\qquad + 0.1\mathrm{E}\left[Y_{t-2}\varepsilon_{t-3}\right] - 0.5\mathrm{E}\left[Y_{t-2}\varepsilon_{t-4}\right] - 0.2\mathrm{E}\left[Y_{t-2}\varepsilon_{t-5}\right] \\
&= 1.3\gamma_1 - 0.4\gamma_0 - 1.332 \qquad (3) \\
\gamma_3 &= 1.3\mathrm{E}\left[Y_{t-3}Y_{t-2}\right] - 0.4\mathrm{E}\left[Y_{t-3}Y_{t-2}\right] \\
&\qquad + 0.1\mathrm{E}\left[Y_{t-3}\varepsilon_{t-3}\right] - 0.5\mathrm{E}\left[Y_{t-3}\varepsilon_{t-4}\right] - 0.2\mathrm{E}\left[Y_{t-3}\varepsilon_{t-5}\right] \\
&= 1.3\gamma_2 - 0.4\gamma_1 - 1.34 \qquad (4) \\
\gamma_4 &= 1.3\mathrm{E}\left[Y_{t-4}Y_{t-2}\right] - 0.4\mathrm{E}\left[Y_{t-4}Y_{t-2}\right] + -0.5\mathrm{E}\left[Y_{t-4}\varepsilon_{t-4}\right] - 0.2\mathrm{E}\left[Y_{t-4}\varepsilon_{t-5}\right] \\
&= 1.3\gamma_3 - 0.4\gamma_2 - 0.9 \qquad (5) \\
\gamma_5 &= 1.3\mathrm{E}\left[Y_{t-5}Y_{t-2}\right] - 0.4\mathrm{E}\left[Y_{t-5}Y_{t-2}\right] + -0.2\mathrm{E}\left[Y_{t-5}\varepsilon_{t-5}\right] \\
&= 1.3\gamma_4 - 0.4\gamma_3 - 0.2. \qquad (6)
\end{aligned}$$

Using the first three equations, it is simple to solve for the first three autocovariances of the process, which are reported in the table below. We then use those values in conjunction with Equations (4) – (6) to obtain the three subsequent autocovariances, which are also reported in the table. Beyond lag 5, Y_t is no longer correlated with ε_t and hence, the process behaves as an $AR(2)$:

$$\gamma_s = 1.3\gamma_{s-1} - 0.4\gamma_{s-2}. \tag{7}$$

Thus, for $s > 5$, γ_s is computed by iterating on Equation (7), using the values in the table below to start the recursions. The first 50 autocovariances are depicted in panel (a) of the figure below.

γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
17.5170	15.9570	12.4010	8.3985	5.0576	3.0155



(d) (7 points) Calculate and plot the first 50 autocorrelations for Y_t.

Solution: The first 50 autocorrelations, ρ_j, are computed via the definition $\rho_j = \frac{\gamma_j}{\gamma_0}$ and are depicted in panel (b) of the figure above.

(e) (7 points) Use R or Python to simulate $n = 1000$ values of Y_t. Do this without using any specialty time series functions (for example, do not use the `arima` function in R). What are the sample mean and variance of your simulation? What are your estimates of the first five autocovariances? How do these values compare with their theoretical counterparts computed in part (c)?

Solution: See solution for part (f).

(f) (7 points) Repeat part (e) for $n = \{10000, 100000, 1000000\}$. How do your estimates of mean, variance and first five autocovariances compare with each other and with the true values that you have already computed?

Solution: The following code simulates the ARMA(2,5) process for different values of n and computes the means, variance and autocovariances for each simulation.

```
# Parameters
phi = c(1.3, -0.4)
theta = c(1, 0.7, 0, 0.1, -0.5, -0.2)

# Function to simulate and compute moments
prob3Sim = function(nSim, phi, theta){
eps = rnorm(nSim+5)
Y = rep(0,nSim+5)
for(i in 6:length(Y)){
Y[i] = phi%*%Y[(i-1):(i-2)] + theta%*%eps[i:(i-5)]
}
Y = Y[-(1:5)]
muHat = signif(mean(Y),4)
gammaHat = signif(acf(Y,plot=FALSE,lag.max=5,type="covariance")$acf[,,1],4)
return(list('mu'=muHat, 'gamma'=gammaHat))
}
```

```
# Simulate for different N
prob3Sim(1000,phi,theta)
prob3Sim(10000,phi,theta)
prob3Sim(100000,phi,theta)
prob3Sim(1000000,phi,theta)
```

The resulting estimates are reported in the table below. As n increases, the estimates are converging to the true values.

n	$\hat{\mu}$	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	$\hat{\gamma}_5$
1000	0.5469	15.49	14.06	10.91	7.467	4.701	3.191
10,000	0.1206	18.18	16.61	13.07	9.116	5.851	3.881
100,000	-0.009310	17.39	15.80	12.22	8.193	4.834	2.788
1,000,000	-0.002353	17.52	15.95	12.39	8.382	5.034	2.987