Econ 211C Midterm Solutions

Code

Instructions

1. You may not discuss this exam with any other person in any way ("discuss" includes any form of electronic communication).
2. You may only reference course materials: notes, textbook and files posted on the Econ 211C website. You may not use Wikipedia, Google or any other online or physical reference.
3. Print (do not sign) your name below. By doing this you pledge to obey and follow the UCSC Academic Integrity policy and to abide by the instructions above.
4. Include this cover sheet (with your name printed below) with your solutions.

Question 1

Suppose that $X_1, \dots, X_n \overset{i.i.d.}{\sim} Poisson(\lambda)$. That is

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!},$$

for $x \in \{0, 1, 2, \dots\}$.

a. (15 points)

Derive the log likelihood function.

Solution:

The likelihood function is

$$\begin{aligned}
\mathcal{L}(\lambda | x_1, \dots, x_n) &= f(x_1, \dots, x_n | \lambda) \\
&= \prod_{i=1}^{n} f(x_i | \lambda) \\
&= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\
&= \lambda^{\sum_{i=1}^{n} x_i} e^{-\lambda n} \prod_{i=1}^{n} \frac{1}{x_i!}.
\end{aligned}$$

The resulting log likelihood is:

$$\ell(\lambda | x_1, \dots, x_n) = \log(\lambda) \sum_{i=1}^{n} x_i - \lambda n - \sum_{i=1}^{n} \log(x_i!).$$

b. (15 points)

Derive the MLE of λ.

Solution:

The MLE, $\hat{\lambda}$, satisfies the first-order condition:

$$\left.\frac{d\ell}{d\lambda}\right|_{\hat{\lambda}} = \frac{\sum_{i=1}^{n} x_i}{\hat{\lambda}} - n = 0$$
$$\Rightarrow \hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

To make sure we actually maximized the log-likelihood (instead of minimizing it), we check that the second derivative is negative:

$$\frac{d^2\ell}{d\lambda^2} = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0.$$

For the Poisson distribution, $\lambda > 0$. Thus, since $\lambda \neq 0$, we know the second derivative exists.

c. (15 points)

State a Central Limit Theorem for the MLE, $\hat{\lambda}$. You may assume that $E[X_i] = \lambda$ for $i = 1, \ldots, n$.

Solution:

The generic Central Limit Theorem for an MLE is

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow{d} N\left(0, n\mathcal{I}(\lambda)^{-1}\right),$$

where $\mathcal{I}(\lambda)$ is the Fisher information for λ:

$$\mathcal{I}(\lambda) = -E\left[\frac{\partial^2\ell}{\partial\lambda^2}\right] = E\left[\frac{\sum_{i=1}^{n} X_i}{\lambda^2}\right] = \frac{n}{\lambda^2}E[X] = \frac{n}{\lambda}.$$

Thus, in this case the Central Limit Theorem can be stated as

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow{d} N(0, \lambda),$$

d. (20 points)

Simulate 1000 $i.i.d.$ samples of size $n = 10$ from a $Poisson(\lambda = 4)$ distribution. Compute the MLE for each sample and report the fraction of estimates that fall within a two standard deviation interval of the true parameter, under the assumption that the limiting distribution in part (c) holds.

Solution:

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```
nMC = 1000
n = 10
lambda = 4
lambdaHat = rep(0,nMC)
for(i in 1:nMC){
  x = rpois(n,lambda)
  lambdaHat[i] = mean(x)
}
sum(lambdaHat>=lambda-2*sqrt(lambda/n) & lambdaHat<=lambda+2*sqrt(lambda/n))
```

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[1] 941
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The simulation above reports 941 estimates falling within a two standard deviation interval.

Question 2

Consider an $AR(1)$ process

$$Y_t = 0.77Y_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \overset{i.i.d.}{\sim} WN(0, \sigma = 2.3)$.

a. (10 points)

Write a formula for the MSE of an s-step ahead forecast. Plot the function as a simple line plot for $s = 1, \dots, 30$.

Solution:

The $AR(1)$ can be expressed as an $MA(\infty)$, where the coefficients of the infinite-order MA lag polynomial, $\psi(L)$, are $\psi_i = \phi^i$. We showed in lecture that the MSE of an s-step forecast of an $MA(\infty)$ is

$$MSE\left(\hat{Y_{t+s|t}}\right) = \sigma^2 \sum_{i=0}^{s-1} \psi_i^2,$$

which in the case the $AR(1)$ specializes to

$$MSE\left(\hat{Y_{t+5|t}}\right) = \sigma^2 \sum_{i=0}^{4} \phi^{2i} = 2.3^2 \sum_{i=0}^{s} 0.77^{2i}.$$

The theoretical MSE for $s = 1, \dots, 30$ is depicted as the solid line in the graph for part (b).

b. (25 points)

Simulate 1000 sample paths of $n = 130$ observations of the $AR(1)$ process. For each simulation, use the 100th observation to make forecasts for $s = 1, \dots, 30$ and compute the errors relative to the true (simulated) observations. Average the squared errors over the 1000 simulations for each value of s, and plot the empirical mean squared error on the same plot as the theoretical MSE computed in part (a).

Solution:

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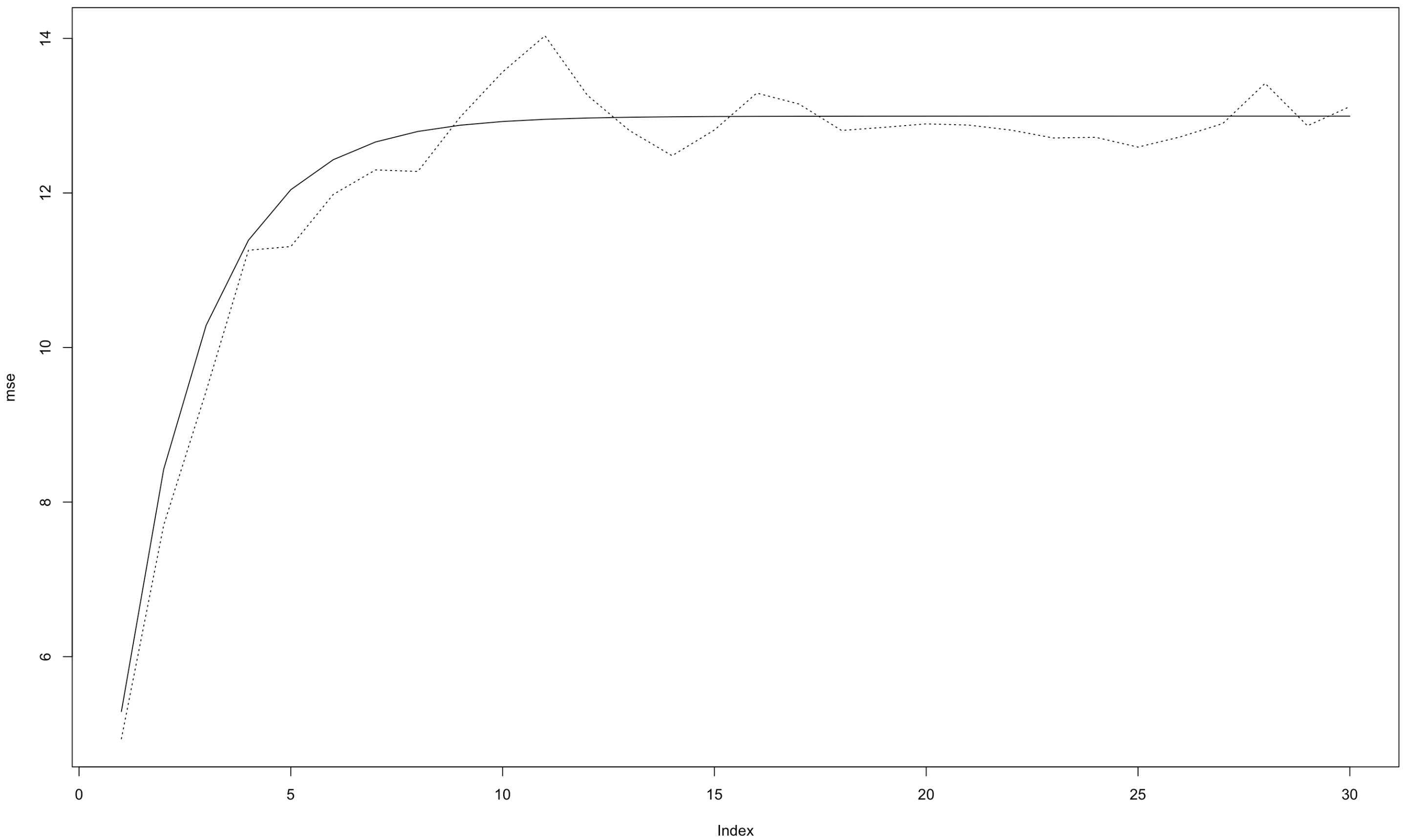
```
n = 100
nForecast = 30
nTotal = n+nForecast
nMC = 1000
phi = 0.77
sigma = 2.3
errorMat = matrix(0,nForecast,nMC)
for(i in 1:nMC){
  y =rep(0,nTotal)
  eps = rnorm(nTotal,0,sigma)
  for(j in 2:nTotal){
    y[j] = phi*y[j-1]+eps[j]
  }
  yForecast = (phi^(1:nForecast))*y[n]
  errorMat[,i] = (yForecast-y[(n+1):(nTotal)])^2
}
mseHat = apply(errorMat,1,mean)
mse = (sigma^2)*cumsum(phi^(2*(0:(nForecast-1))))
yMin = min(min(mse),min(mseHat))
yMax = max(max(mse),max(mseHat))
plot(mse,type='l',ylim=c(yMin,yMax))
lines(mseHat,lty=3)
```



mse
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