ECON 205C: Problem Set 1

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Problem 1

(a)

The Euler condition, $C_t^{-\sigma} = \beta E_t(\frac{1+i_t}{1+\pi_{t+1}})C_{t+1}^{-\sigma}$, can be linearized as

$$
\begin{aligned}
\bar{C}^{-\sigma}(1-\sigma\hat{c}_t) &= \beta E_t[1+i_t-\pi_{t+1}]\bar{C}^{-\sigma}(1-\sigma\hat{c}_{t+1}) \\
\Leftrightarrow \ (1-\sigma\hat{c}_t) &= \Big(\frac{1}{1+r}\Big)E_t[1+i_t-\pi_{t+1}](1-\sigma\hat{c}_{t+1}) \\
&\approx E_t[1+i_t-\pi_{t+1}-r](1-\sigma\hat{c}_{t+1}) \\
&\approx E_t[1+i_t-\pi_{t+1}-r-\sigma\hat{c}_{t+1}) \\
\Leftrightarrow \ -\sigma\hat{c}_t &= E_t[i_t-\pi_{t+1}-r-\sigma\hat{c}_{t+1}) \\
\Leftrightarrow \ \hat{c}_t &= E_t[-\frac{1}{\sigma}\Big(i_t-\pi_{t+1}-r\Big)+\hat{c}_{t+1}]
\end{aligned}
$$

$$\therefore \quad \hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{\sigma}\big(i_t - r - E_t\pi_{t+1}\big).$$

(b)

We consider two cases of habit persistences that households show.

i. External habit persistence:

By treating C_{t-1} exogenous to the individual, the first order condition for optimal condition becomes

$$\lambda_t = (C_t - hC_{t-1})^{-\sigma}$$

and the Euler condition becomes

$$U'(C_t) = \beta E_t\Big(\frac{1+i_t}{1+\pi_{t+1}}\Big)U'(C_{t+1})$$

$$\Leftrightarrow (C_t - hC_{t-1})^{-\sigma} = \beta E_t\Big(\frac{1+i_t}{1+\pi_{t+1}}\Big)(C_{t+1} - hC_t)^{-\sigma}.$$

ii. Internal habit persistence:

In the case of C_{t-1} being endogenous, the first order condition for optimal condition becomes

$$\lambda_t = (C_t - hC_{t-1})^{-\sigma} - \beta h(C_{t+1} - hC_t)^{-\sigma}$$

and the Euler condition takes a slightly different form as follows

$$U'(C_t) - \beta E_t U'(C_{t+1}) = \beta E_t\Big(\frac{1+i_t}{1+\pi_{t+1}}\Big)\big(U'(C_{t+1}) - \beta E_{t+1}U'(C_{t+2})\big)$$

$$\Leftrightarrow (C_t - hC_{t-1})^{-\sigma} - \beta E_t(C_{t+1} - hC_t)^{-\sigma} = \beta E_t\Big(\frac{1+i_t}{1+\pi_{t+1}}\Big)\big(C_{t+1} - hC_t)^{-\sigma} - \beta E_{t+1}(C_{t+2} - hC_{t+1})^{-\sigma}\big)$$

(c)

To derive the linearized Euler condition for the case of part (b) i., we first need to linearize the marginal utility of consumption.

$$(C_t - hC_{t-1})^{-\sigma} \approx \bar{C}^{-\sigma}(1 - \hat{\xi}_t)$$
$$\xi_t \equiv (C_t - hC_{t-1})^{-\sigma}$$

Take logarithm on both sides.

$$\log \xi_t = -\sigma \log(C_t - hC_{t-1})$$

Take total differential at steady-state value.

$$\begin{aligned}
\frac{1}{\xi} d\xi_t &= -\sigma \frac{1}{C - hC}(dC_t - h \cdot dC_{t-1}) \\
\hat{\xi}_t &\approx -\sigma \frac{1}{1-h}\frac{1}{C}(dC_t - h \cdot dC_{t-1}) \\
\hat{\xi}_t &\approx -\sigma \frac{1}{1-h}(\hat{c}_t - h\hat{c}_{t-1}) \\
\therefore \quad (C_t - hC_{t-1})^{-\sigma} &\approx \bar{C}^{-\sigma}(1 - \sigma \frac{1}{1-h}(\hat{c}_t - h\hat{c}_{t-1}))
\end{aligned}$$

Now, let's derive the linearized Euler condition.

$$\begin{aligned}
\bar{C}^{-\sigma}(1 - \sigma\frac{1}{1-h}(\hat{c}_t - h\hat{c}_{t-1})) &\approx E_t(1 + i_t - \pi_{t+1} - \rho)\bar{C}^{-\sigma}(1 - \sigma\frac{1}{1-h}(\hat{c}_{t+1} - h\hat{c}_t)) \\
(1 - \sigma\frac{1}{1-h}(\hat{c}_t - h\hat{c}_{t-1})) &\approx E_t(1 + i_t - \pi_{t+1} - \rho)(1 - \sigma\frac{1}{1-h}(\hat{c}_{t+1} - h\hat{c}_t)) \\
(1 - \sigma\frac{1}{1-h}(\hat{c}_t - h\hat{c}_{t-1})) &\approx E_t(1 + i_t - \pi_{t+1} - \rho - \sigma\frac{1}{1-h}(\hat{c}_{t+1} - h\hat{c}_t)) \\
\hat{c}_t - h\hat{c}_{t-1} &\approx E_t\hat{c}_{t+1} - h\hat{c}_t - \frac{1}{\sigma}(1-h)(i_t - \pi_{t+1} - \rho - \sigma\frac{1}{1-h}) \\
\therefore \quad \hat{c}_t &\approx \frac{1}{1+h}E_t\hat{c}_{t+1} + \frac{h}{1+h}\hat{c}_{t-1} - \frac{(1-h)}{\sigma(1+h)}(i_t - \pi_{t+1} - \rho - \sigma\frac{1}{1-h})
\end{aligned}$$

let $x_t = \hat{c}_t - h\hat{c}_{t-1}$

$\Rightarrow -\frac{\sigma}{1-h}x_t = E_t\Big[i_t - \pi_{t+1} - \rho - \sigma\frac{1}{1-h}x_{t+1}$

after subtraction 1 from each side

We can see that $\hat{c}_{t-1}$ and h is in the Euler condition. That means past consumption is considered for the optimal intertemporal allocation of consumption. As h is larger, it means that past consumption has a larger influence in current utility of consumption.

Problem 2

(a)

The given NK model can be rearranged into a system of two equations:

$$E_t x_{t+1} + \sigma^{-1}E_t\pi_{t+1} = x_t + \sigma^{-1}\phi\pi_t - \sigma^{-1}r_t$$
$$\beta E_t\pi_{t+1} = -\kappa x_t + \pi_t - u_t.$$

In matrix form:

$$\begin{bmatrix}1 & \sigma^{-1}\\ 0 & \beta\end{bmatrix}\begin{bmatrix}E_t x_{t+1}\\ E_t\pi_{t+1}\end{bmatrix} = \begin{bmatrix}1 & \sigma^{-1}\phi\\ -\kappa & 1\end{bmatrix}\begin{bmatrix}x_t\\ \pi_t\end{bmatrix} + \begin{bmatrix}-\sigma^{-1} & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}r_t\\ u_t\end{bmatrix}$$

$$\begin{aligned}
\Leftrightarrow \begin{bmatrix}E_t x_{t+1}\\ E_t\pi_{t+1}\end{bmatrix} &= \begin{bmatrix}1 & \sigma^{-1}\\ 0 & \beta\end{bmatrix}^{-1}\begin{bmatrix}1 & \sigma^{-1}\phi\\ -\kappa & 1\end{bmatrix}\begin{bmatrix}x_t\\ \pi_t\end{bmatrix} + \begin{bmatrix}1 & \sigma^{-1}\\ 0 & \beta\end{bmatrix}^{-1}\begin{bmatrix}-\sigma^{-1} & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}r_t\\ u_t\end{bmatrix}\\
&= \frac{1}{\beta}\begin{bmatrix}\beta & -\sigma^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \sigma^{-1}\phi\\ -\kappa & 1\end{bmatrix}\begin{bmatrix}x_t\\ \pi_t\end{bmatrix} + \frac{1}{\beta}\begin{bmatrix}\beta & -\sigma^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}-\sigma^{-1} & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}r_t\\ u_t\end{bmatrix}\\
&= \frac{1}{\beta}\begin{bmatrix}\beta+\sigma^{-1}\kappa & \beta\sigma^{-1}\phi-\sigma^{-1}\\ -\kappa & 1\end{bmatrix}\begin{bmatrix}x_t\\ \pi_t\end{bmatrix} + \frac{1}{\beta}\begin{bmatrix}-\beta\sigma^{-1} & \sigma^{-1}\\ 0 & -1\end{bmatrix}\begin{bmatrix}r_t\\ u_t\end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
M &= \begin{bmatrix}1+\frac{\kappa}{\beta\sigma} & \frac{\phi}{\sigma}-\frac{1}{\beta\sigma}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}\\
N &= \begin{bmatrix}-\frac{1}{\sigma} & \frac{1}{\beta\sigma}\\ 0 & -\frac{1}{\beta}\end{bmatrix}
\end{aligned}$$

(b)

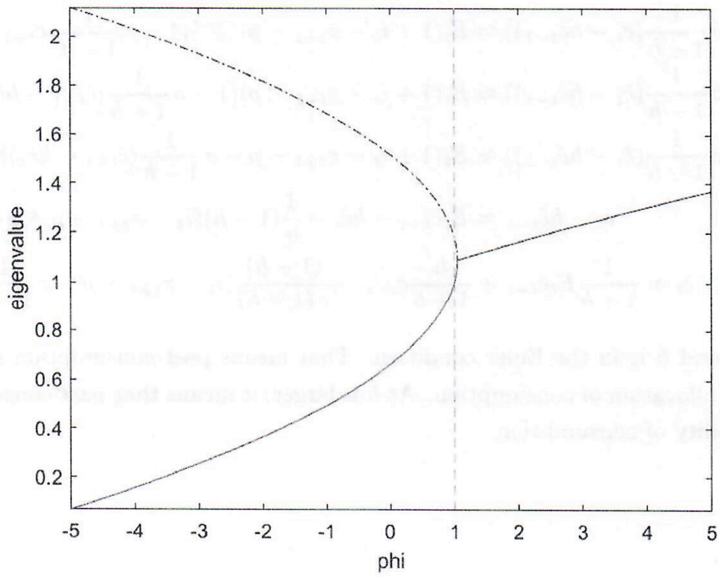

Figure 1: The absolute values of the two eigenvalues of M as functions of ϕ

The above figure shows that for $\phi > 1$ both eigenvalues lie outside of the unit circle (i.e. $|\lambda_i| > 1$ for i=1,2), and thus satisfying the Blanchard-Khan conditions.

(c)

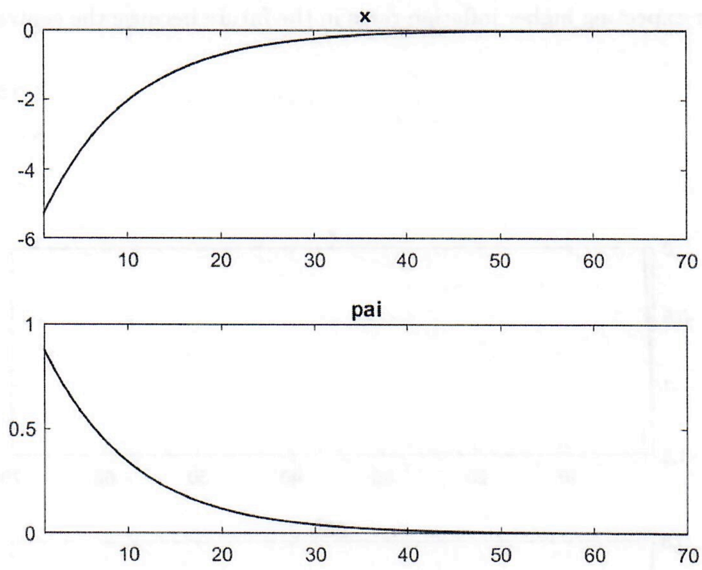

Figure 2: The responses of the output gap and inflation to a positive realization of v_t with $\phi = 1.5$

(d)

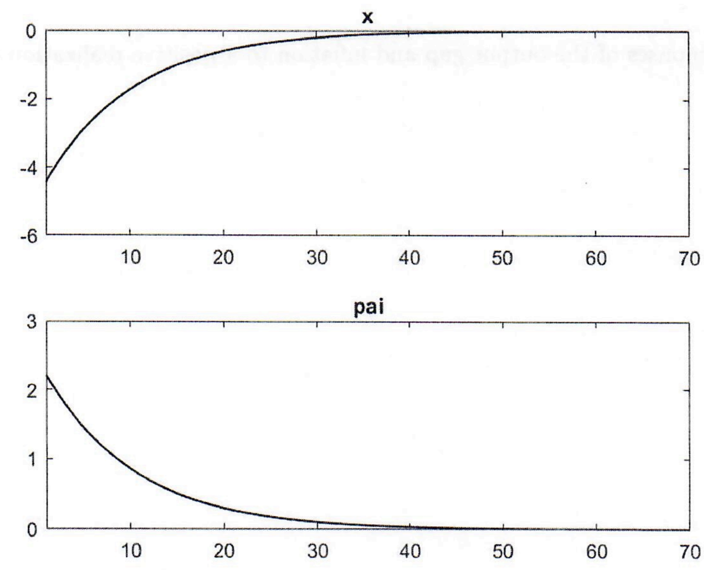

Figure 3: The responses of the output gap and inflation to a positive realization of v_t with $\phi = 1.1$

A less sensitive policy would lead to lower nominal interest rates which would increase demand from the consumer perspective, thus increasing output and shrinking the output gap (as it is negative in this model), which we can see in the graphs for parts (c) and (d). The initial jump being larger for a less sensitive policy is due to the consumer expecting higher inflation rates in the future because the central bank has less control over inflation rates.

CB responds less to inflation so it stabilizes inflation less.

(e)

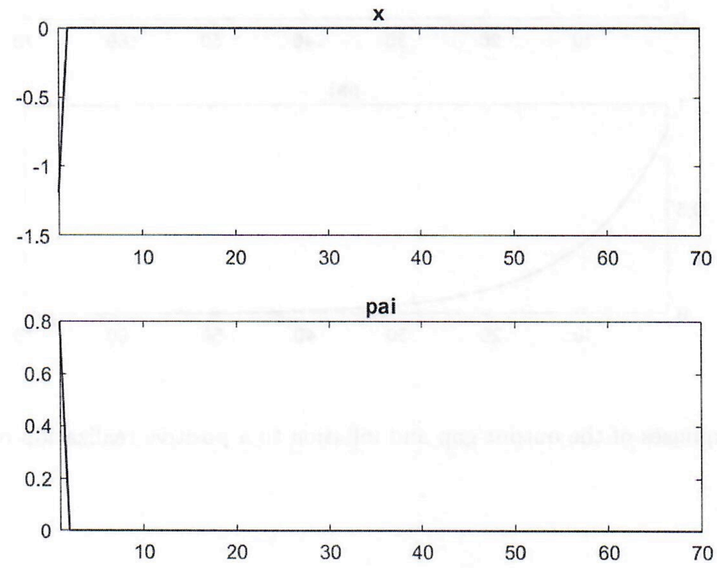

Figure 4: The responses of the output gap and inflation to a positive realization of u_t with $\phi = 1.5$

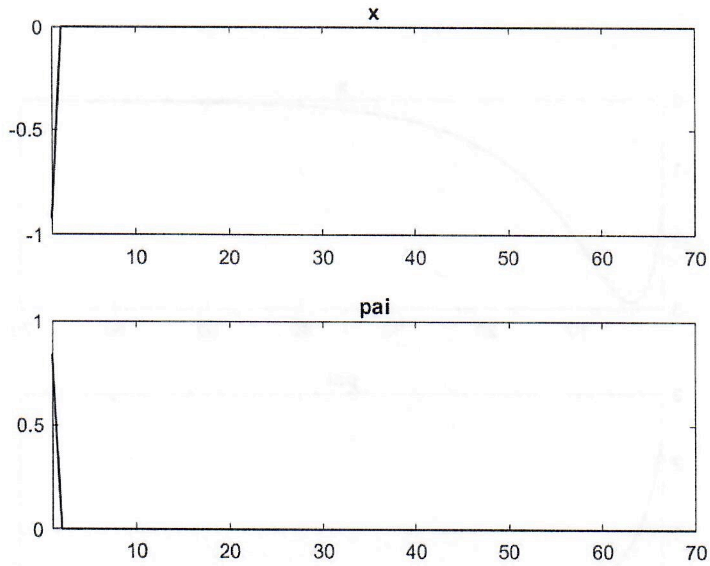

Figure 5: The responses of the output gap and inflation to a positive realization of u_t with $\phi = 1.1$

?

The impacts of cost shock on u_t on output gap and inflation rate are not persistent, unlike in the case of inflation shocks. That is, after the shock at the first period the values converge to the steady state just after one period. Since no expectation term is involved, the magnitude of the jumps are the same.

u_t has a AR(1) coeff of 0.9.

Problem 3

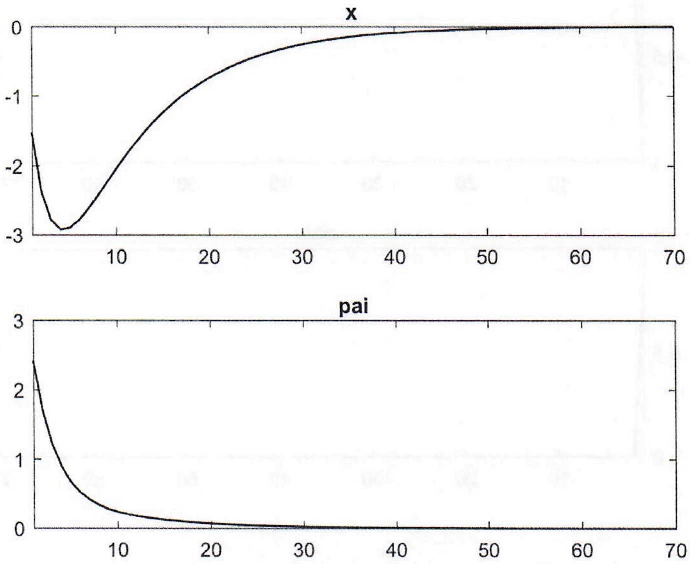

Figure 6: The responses of the output gap and inflation to a positive realization of u_t with $\phi = 1.5$

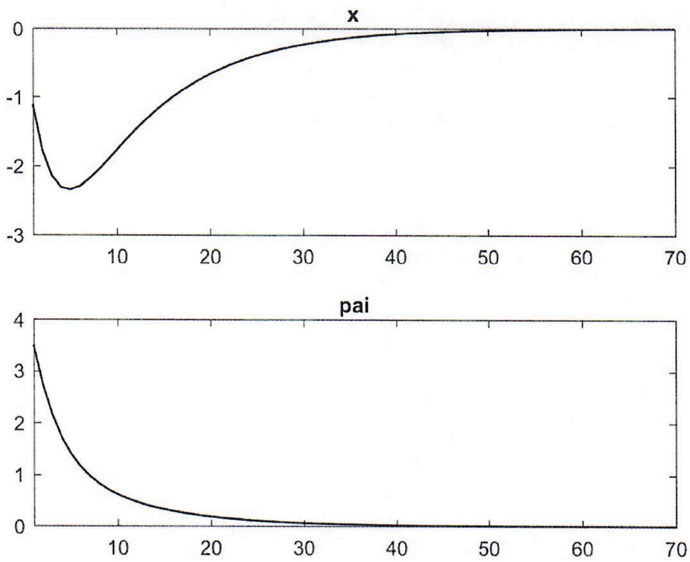

Figure 7: The responses of the output gap and inflation to a positive realization of u_t with $\phi = 1.1$

Unlike in part (c) of Problem 2, the output gap includes lag effect from the past. Due to the inertia built in the system, we see that the response of the output graph looks different from what we obtained before.

We can confirm that the output gap initially decreases and starts to converge to the steady state from time $t+5$. On the other hand, for the same reason explained above, the magnitude of jumps differs with different levels of ϕ.

Problem 4

(a)

The optimization problem of the central bank's optimization problem under discretion at every time period is

$$\min_{\pi_t, x_t, i_t} \frac{1}{2}[\pi_t^2 + \lambda_x x_t^2 + \lambda_i (i_t - i^*)^2] \quad \text{s.t.} \quad \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + e_t$$
$$x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t).$$

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\pi_t^2 + \lambda_x x_t^2 + \lambda_i (i_t - i^*)^2) + \psi_t(\pi_t - \beta E_t \pi_{t+1} - \kappa x_t - e_t) + \theta_t(x_t - E_t x_{t+1} + \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t)).$$

The first order necessary conditions are

$$[\pi_t] \quad \pi_t + \psi_t = 0$$
$$[x_t] \quad \lambda_x x_t - \psi_t \kappa + \theta_t = 0$$
$$[i_t] \quad \lambda_i (i_t - i^*) + \frac{\theta_t}{\sigma} = 0.$$

(b)

We can see from part (a) by rearranging the FOC for i_t that we get the relationship $\theta_t = -\sigma \lambda_i (i_t - i^*)$, showing that θ_t will be non-zero as long as $\lambda_i \neq 0$ (and thus if $\lambda_i > 0$). Intuitively, θ_t represents the change in the objective (loss) function relative to interest (by Envelope Theorem) and λ_i is the weight placed on nominal interest relative to inflation in the loss function. Any change in interest rates would affect the objective (loss) function and thus affecting the choice variable (i_t) of the central bank. So, if the central bank places a non-zero (positive) weight on nominal interest rates in the loss function, the change in the loss function with respect to nominal interest rates would be non-negligible

See answer key.

(c)

The central bank's optimization problem under full commitment is

$$\min_{\{\pi_s, x_s, i_s\}_{s=t}^{\infty}} \frac{1}{2} E_t \sum_{i=0}^{\infty} \beta^i [\pi_{t+i}^2 + \lambda_x x_{t+i}^2 + \lambda_i (i_{t+i} - i^*)^2] \quad \text{s.t.} \quad \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + e_t$$
$$x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t).$$

The Lagrangian is

$$\mathcal{L} = E_t \sum_{i=0}^{\infty} \beta^i \Big\{ \frac{1}{2}[\pi_{t+i}^2 + \lambda_x x_{t+i}^2 + \lambda_i (i_{t+i} - i^*)^2] + \psi_{t+i}(\pi_{t+i} - \beta\pi_{t+1+i} - \kappa x_{t+i} - e_{t+i})$$

$$+\theta_{t+i}(x_{t+i} - E_t x_{t+1+i} + \frac{1}{\sigma}(i_{t+i} - \pi_{t+1+i} - r_{t+i}))\Big\}.$$

The first order necessary conditions are

$[\pi_t]$ $\boxed{\pi_t + \psi_t = 0}$

$[x_t]$ $\boxed{\lambda_x x_t - \psi_t \kappa + \theta_t = 0}$

$[\pi_{t+i}]$ $\beta^i(\pi_{t+i} + \psi_{t+i}) + \beta^{i-1}(-\beta\psi_{t-1+i} - \frac{1}{\sigma}\theta_{t-1+i}) = 0 \quad \Leftrightarrow \quad \boxed{\pi_{t+i} + \psi_{t+i} - \psi_{t-1+i} - \frac{1}{\beta\sigma}\theta_{t-1+i} = 0}$

$[x_{t+i}]$ $\beta^i(\lambda_x x_{t+i} - \psi_{t+i}\kappa + \theta_{t+i}) - \beta^{i-1}(\theta_{t-1+i}) = 0 \quad \Leftrightarrow \quad \boxed{\lambda_x x_{t+i} - \psi_{t+i} + \theta_{t+i}\kappa - \frac{1}{\beta}\theta_{t-1+i} = 0}$

$[i_{t+i}]$ $\boxed{\lambda_i(i_{t+i} - i^*) + \frac{1}{\sigma}\theta_{t+i} = 0}$

where the first two conditions are for $i = 0$ and the last three are for $i > 0$.
Note that for $i = 0$, the conditions are the same as in part (a).

(d)

From a timeless perspective, there is no separate conditions for $i = 0$. That is, the conditions for π_t and x_t in part (c) are no longer considered. Therefore, the system of first order conditions becomes

$$\pi_{t+i} + \psi_{t+i} - \psi_{t-1+i} - \frac{1}{\beta\sigma}\theta_{t-1+i} = 0 \tag{1}$$

$$\lambda_x x_{t+i} - \psi_{t+i}\kappa + \theta_{t+i} - \frac{1}{\beta}\theta_{t-1+i} = 0 \tag{2}$$

$$\lambda_i(i_{t+i} - i^*) + \frac{1}{\sigma}\theta_{t+i} = 0, \tag{3}$$

which should be satisfied for all $i \geq 0$.
Equation (3) can be simplified as

$$\theta_{t+i} = -\sigma\lambda_i(i_t - i^*).$$

By inserting this into equation (2), we get

$$\lambda_x x_{t+i} - \psi_{t+i}\kappa - \sigma\lambda_i(i_{t+1} - i^*) + \frac{\sigma}{\beta}\lambda_i(i_{t-1+i} - i^*) = 0$$

$$\Leftrightarrow \psi_{t+i} = \frac{1}{\kappa}\Big(\lambda_x x_{t+i} - \sigma\lambda_i(i_{t+1} - i^*) + \frac{\sigma}{\beta}\lambda_i(i_{t-1+i} - i^*)\Big).$$

By inserting this into equation (1), we get

$$\begin{aligned}
\pi_{t+i} &= -\psi_{t+i} + \psi_{t-1+i} + \frac{1}{\beta\sigma}\theta_{t-1+i} \\
&= -\frac{1}{\kappa}\left(\lambda_x x_{t+i} - \sigma\lambda_i(i_{t+1} - i^*) + \frac{\sigma}{\beta}\lambda_i(i_{t-1+i} - i^*)\right) \\
&\quad +\frac{1}{\kappa}\left(\lambda_x x_{t+i-1} - \sigma\lambda_i(i_t - i^*) + \frac{\sigma}{\beta}\lambda_i(i_{t-2+i} - i^*)\right) + \frac{1}{\beta\sigma}\theta_{t-1+i}.
\end{aligned}$$

Problem 5

The optimal commitment policy in a basic NK model must abide by the basic structure of commitment, which is that the promises made at the beginning time period t must be fulfilled. The past commitments that have been made and fulfilled, and thus future expectations must influence the central bank's choices today. Hence today's decision is not just forward-looking but also takes the lagged value of output gap into account to ensure that past promises/commitments are not ignored.

Can you express this more clearly?

not clear how what follows after "Hence" was implied by first 2 sentences.