Homework 3 Answer Key

Econ 205B, Winter 2017

1. (a) Let the value function be

$$V(\omega_t, m_t) = \max\{U(c_t, m_t) + \beta V(\omega_{t+1}, m_{t+1})\}$$

subject to

$$\omega_{t+1} = f(k_t) + (1-\delta)k_t + (1+r_t)b_t + m_{t+1}$$

and

$$\omega_t - c_t - m_{t+1}(1+\pi_{t+1}) - b_t - k_t = 0.$$

In this setup, r_t is the real return on bonds. Let λ_t denote the Lagrangian on this last constraint. First order conditions for c_t, m_{t+1}, b_t, and k_t plus the envelope theorem give

$$\begin{aligned} U_c(c_t, m_t) &= \lambda_t \\ \beta V_m(\omega_{t+1}, m_{t+1}) + \beta V_\omega(\omega_{t+1}, m_{t+1}) &= \lambda_t(1+\pi_{t+1}) \\ \beta(1+r_t)V_\omega(\omega_{t+1}, m_{t+1}) &= \lambda_t \\ \beta V_\omega(\omega_{t+1}, m_{t+1}) &= \lambda_t \\ V_m(\omega_t, m_t) &= U_m(c_t, m_t). \end{aligned}$$

Now combining the second and third of these to obtain

$$\beta V_m(\omega_{t+1}, m_{t+1}) + \beta V_\omega(\omega_{t+1}, m_{t+1}) = \beta(1+\pi_{t+1})(1+r_t)V_\omega(\omega_{t+1}, m_{t+1})$$

or

$$\begin{aligned} \beta V_m(\omega_{t+1}, m_{t+1}) &= [(1+\pi_{t+1})(1+r_t) - 1]\beta V_\omega(\omega_{t+1}, m_{t+1}) \\ &= i_t \beta V_\omega(\omega_{t+1}, m_{t+1}) \end{aligned}$$

Hence,

$$\frac{V_m(\omega_{t+1}, m_{t+1})}{V_\omega(\omega_{t+1}, m_{t+1})} = i_t.$$

But the last of the first order conditions implies

$$V_m(\omega_{t+1}, m_{t+1}) = U_m(c_{t+1}, m_{t+1})$$

while the first and the fourth yield

$$V_\omega(\omega_{t+1}, m_{t+1}) = \lambda_{t+1} = U_c(c_{t+1}, m_{t+1})$$

Thus,

$$\frac{U_m(c_{t+1}, m_{t+1})}{U_c(c_{t+1}, m_{t+1})} = i_t.$$

(b) In the case considered in class, cash held in time t yielded utility at time t but the opportunity cost was in the lost interest income from the bonds that could otherwise have been held. Since this interest payment occurs in $t+1$, it must be discounted back to compare to the marginal value from holding money at time t.

2. (a) The household's problem is to maximize

$$\sum_{t=0}^{\infty} \beta^t u(C_t^m, C_t^c)$$

for $0 < \beta < 1$, subject to a sequence of CIA constraints of the form, in real terms

$$C_t^m \leq \frac{M_{t-1}}{P_t} + \frac{T_t}{P_t} = \frac{m_{t-1}}{\Pi_t} + \tau_t$$

where $m_{t-1} = M_{t-1}/P_{t-1}$, $\Pi_t = P_t/P_{t-1}$ is 1 plus the inflation rate, and $\tau_t = T_t/P_t$, and a sequence of budget constraints of the form, also in real terms,

$$Y_t + \tau_t + \frac{m_{t-1} + I_{t-1}b_{t-1}}{\Pi_t} \geq C_t^m + C_t^c + m_t + b_t + k_t$$

where m and b are real cash holdings. Note that real resources available to the representative agent in period $t+1$ are given by

$$Y_{t+1} + \tau_{t+1} + \frac{m_t + I_t b_t}{\Pi_t}.$$

The period t nominal interest factor I_t divided by Π_{t+1} is the gross real rate of return from period t to $t+1$ and can be denoted by $R_t \equiv I_t/\Pi_{t+1}$.

(b) Let λ_t and μ_t be the time t Lagrangian multipliers on the budget and CIA constraints respectively. Then the necessary FOC for the two types of consumption are

$$
\begin{aligned}
u_{c^m}(t) &= \lambda_t + \mu_t \\
u_{c^c}(t) &= \lambda_t \\
\lambda_t &= \beta(1 - \delta + f')\lambda_{t+1} \\
\lambda_t &= \beta\left(\frac{I_t}{\Pi_{t+1}}\right)\lambda_{t+1} = \beta\left(\frac{1+i_t}{\Pi_{t+1}}\right)\lambda_{t+1} \\
\lambda_t &= \beta\left(\frac{\lambda_{t+1} + \mu_{t+1}}{\Pi_{t+1}}\right)
\end{aligned}
$$

(c) Subtracting the last FOC from the second-to-last one, we derive

$$
\begin{aligned}
\lambda_t - \lambda_t &= \beta\left(\frac{1+i_t}{\Pi_{t+1}}\right)\lambda_{t+1} - \beta\left(\frac{1+i_t}{\Pi_{t+1}}\right)\lambda_{t+1} \\
\implies 0 &= \beta\left(\frac{i_t\lambda_{t+1} - \mu_{t+1}}{\Pi_{t+1}}\right) \\
\implies \mu_{t+1} &= i_t\lambda_{t+1}
\end{aligned}
$$

Hence, the FOC for consumption can be written as

$$
\begin{aligned}
u_{c^m}(t) &= \lambda_t + \mu_t = \lambda_t(1 + i_{t-1}) \\
u_{c^c}(t) &= \lambda_t
\end{aligned}
$$

which shows that i_{t-1} acts as a tax on C_t^m.

3. (a) Cost minimization problem:

$$
\begin{aligned}
&\min_{H_{j,t}, K_{j,t}} \left[\frac{W_t}{P_t}H_{jt} + \frac{R_t^k}{P_t}K_{jt}\right] \\
&\text{s.t.} \qquad z_t K_{j,t}^{\alpha} H_{j,t}^{1-\alpha} \geq Y_{j,t}
\end{aligned}
$$

Let φ_t be the lagrangian multiplier. Then the FONC for $H_{j,t}$ implies

$$
\begin{aligned}
\frac{W_t}{P_t} &= \varphi_t(1-\alpha)\frac{z_t K_{j,t}^{\alpha} H_{j,t}^{1-\alpha}}{H_{j,t}} \\
&= \varphi_t(1-\alpha)z_t\left(\frac{K_{jt}}{H_{jt}}\right)^{\alpha}
\end{aligned}
$$

and the FONC for $K_{j,t}$ implies

$$
\begin{aligned}
\frac{R_t^k}{P_t} &= \varphi_t\alpha\frac{z_t K_{j,t}^{\alpha} H_{j,t}^{1-\alpha}}{K_{j,t}} \\
&= \varphi_t\alpha z_t\left(\frac{K_{jt}}{H_{jt}}\right)^{\alpha-1}
\end{aligned}
$$

Using these two equations, we get

$$mc_t = \varphi_t = \frac{1}{z_t \alpha^\alpha (1-\alpha)^{1-\alpha}} \left(\frac{W_t}{P_t}\right)^{1-\alpha} \left(\frac{R_t^k}{P_t}\right)^{\alpha}$$

(b) Profit maximization problem:

$$\max_{P_{j,t}} E_t \sum_{i=0}^{\infty} \xi^i \beta^i \lambda_{t+i} \left[\frac{P_{j,t} V_{t,i}}{P_{t+i}} - \varphi_{t+i}\right] Y_{j,t+i}$$

where

$$V_{t,l} = \begin{cases} 1 & \text{if} \quad l = 0 \\ \pi_t^\gamma \times \cdots \times \pi_{t+l-1}^\gamma \pi^{(1-\gamma)l} & \text{if} \quad l \geq 0 \end{cases}$$

(c) Plug in the demand function for the intermediate goods:

$$\max_{P_{j,t}} E_t \sum_{i=0}^{\infty} \xi^i \beta^i \lambda_{t+i} \left[\left(\frac{P_{j,t} V_{t,i}}{P_{t+i}}\right)^{1-\theta} - \varphi_{t+i} \left(\frac{P_{j,t} V_{t,i}}{P_{t+i}}\right)^{-\theta}\right] Y_{t+i}$$

The FONC for $P_{j,t} = P_t^*$ is

$$E_t \sum_{i=0}^{\infty} \xi^i \beta^i \lambda_{t+i} \left[(1-\theta)\left(\frac{P_{j,t} V_{t,i}}{P_{t+i}}\right) + \theta \varphi_{t+i}\right] \left(\frac{1}{P_t^*}\right) \left(\frac{P_t^* V_{t,i}}{P_{t+i}}\right)^{-\theta} Y_{t+i} = 0$$

Let $p_t^* \equiv \frac{P_t^*}{P_t}$. Then

$$\begin{aligned} p_t^* &= \frac{\theta}{\theta - 1} \cdot \frac{E_t \sum_{i=0}^{\infty} \xi^i \beta^i \lambda_{t+i} \varphi_{t+i} \left(\frac{P_{t+i}}{P_t V_{t,i}}\right)^{\theta} Y_{t+i}}{E_t \sum_{i=0}^{\infty} \xi^i \beta^i \lambda_{t+i} \left(\frac{P_{t+i}}{P_t V_{t,i}}\right)^{\theta-1} Y_{t+i})} \\ &= \frac{\theta}{\theta-1} \cdot \frac{P_t^n}{P_t^d} \end{aligned}$$

Recursively,

$$\begin{aligned} P_t^n &= \lambda_t \varphi_t Y_t + \xi \beta E_t \left(\frac{\pi_{t+1}}{\pi_t^\gamma \pi^{1-\gamma}}\right)^{\theta} P_{t+1}^n \\ P_t^d &= \lambda_t Y_t + \xi \beta E_t \left(\frac{\pi_{t+1}}{\pi_t^\gamma \pi^{1-\gamma}}\right)^{\theta-1} P_{t+1}^d \end{aligned}$$

Also we have the evolution for the aggregate price index:

$$P_t^{1-\theta} = (1-\xi)(P_t^*)^{1-\theta} + \xi(\pi_{t-1}^\gamma \pi^{1-\gamma} P_{t-1})^{1-\theta}$$

which can be rearranged as

$$1 = (1-\xi)(p_t^*)^{1-\theta} + \xi\left(\frac{\pi_{t-1}^{\gamma}\pi^{1-\gamma}}{\pi_t}\right)^{1-\theta}$$

Log linearizing this equations, we get the NKPC with price indexation:

$$\hat{\pi}_t = \beta \cdot \frac{1}{1+\gamma\beta} E_t\hat{\pi}_{t+1} + \frac{\gamma}{1+\gamma\beta}\hat{\pi}_{t-1} + \frac{(1-\xi\beta)(1-\xi)}{\xi(1+\gamma\beta)}\hat{\varphi}_t$$

where when $\gamma = 0$, it reduces to

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{(1-\xi\beta)(1-\xi)}{\xi}\hat{\varphi}_t$$

(d) As γ increases, inflation rate will be more autocorrelated because it depends more on the lagged inflation rate.

4. (a) The first order condition for the household's choice of N equates the marginal rate of substitution between leisure and consumption to the real wage, or

$$\frac{\xi N_t^{\eta}}{C_t^{-\sigma}} = w_t.$$

A positive realization of ξ_t increases the disutility of work. For a given real wage and marginal utility of consumption, this reduces the household's desired level of time spent working:

$$\frac{\partial N_t}{\partial \xi_t} = -\left(\frac{N_t}{\eta\xi_t}\right) < 0.$$

This is a partial equilibrium effect as both C_t and we will also adjust.

(b) If we linearize the first order condition with respect to the labor supply and note that $\hat{w}_t = z_t$, we obtain

$$\hat{\xi} + \eta\hat{n}_t + \sigma\hat{c}_t = z_t$$

in a flexible-price equilibrium . With goods market clearing requiring that $\hat{c}_t = \hat{y}_t$, and the production function implying $\hat{y}_t = \hat{n}_t + z_t$, we can write this condition as

$$\hat{\xi} + \eta(\hat{y}_t - z_t) + \sigma\hat{y}_t = z_t \implies \hat{y}_t^f = \frac{(1+\eta)z_t - \hat{\xi}_t}{\sigma+\eta}.$$

(c) Yes. Because a positive ξ_t reduces desired labor supply for a given real wage and marginal utility of consumption, it reduces the output in the flexible-price equilibrium.

(d) This Euler condition was obtained from the condition that

$$\hat{c}_t = E_t \hat{c}_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - E_t \pi_{t+1}),$$

then using the fact that $\hat{c}_t = \hat{y}_t$ to yield

$$\hat{y}_t = E_t \hat{y}_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - E_t \pi_{t+1}).$$

Finally, we added and subtracted current and expected future flexible price output to obtain

$$\hat{y}_t - \hat{y}_t^f = E_t \hat{y}_{t+1} - E_t \hat{y}_{t+1}^f - \left(\frac{1}{\sigma}\right)(i_t - E_t \pi_{t+1}) + E_t \hat{y}_{t+1}^f - \hat{y}_t^f,$$

which shows that

$$r_t^n = \sigma\left(E_t \hat{y}_{t+1}^f - \hat{y}_t^f\right).$$

Expected growth in the flexible-price output level increases r_t^n to induce households to willingly have consumption lower today than in the future. Since r_t^n depends on $\hat{y}_t^f$, it depends on $\hat{\xi}_t$. Using the results from part (b),

$$r_t^n = \sigma\left(E_t \hat{y}_{t+1}^f - \hat{y}_t^f\right) = \frac{\sigma(1+\eta)\left(E_t z_{t+1} - z_t\right) - \left(E_t \hat{\xi}_{t+1} - \hat{\xi}_t\right)}{\sigma + \eta}.$$

If $\hat{\xi}_t$ is an AR(1) process, $\hat{\xi}_{t+1} = \rho_\xi \hat{\xi}_t + \epsilon_t^\xi$, then the effect of $\hat{\xi}_t$ on r_t^n is given by

$$\left(\frac{\sigma}{\sigma + \eta}\right)(1 - \rho_\xi) > 0.$$

A positive $\hat{\xi}_t$ temporarily lowers the flexible price output level at time t relative to its value at $t+1$. To induce households to reduce current consumption relative to future consumption, the real interest rate must rise.