Homework 2 Answer Key

Econ 205B, Winter 2017

1. (a)

$$
\begin{aligned}
&\frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}}(R_{t+1}+1-\delta)\right] \\
&\varphi H_t^\eta = \frac{W_t}{C_t} \\
&Y_t = z_t K_{t-1}^\alpha H_t^{1-\alpha} \\
&W_t = (1-\alpha)\frac{Y_t}{H_t} \\
&R_t = \alpha\frac{Y_t}{K_{t-1}} \\
&K_t = (1-\delta)K_{t-1} + I_t \\
&C_t + I_t + G_t = Y_t
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}}(R_{t+1}+1-\delta)\right] \\
&\varphi H_t^\eta = \frac{W_t}{C_t} \\
&Y_t = z_t K_{t-1}^\alpha H_t^{1-\alpha} \\
&W_t = (1-\alpha)\frac{Y_t}{H_t} \\
&R_t = \alpha\frac{Y_t}{K_{t-1}} \\
&K_t = (1-\delta)K_{t-1} + I_t + G_t \\
&C_t + I_t + G_t = Y_t
\end{aligned}
$$

(c) The government spending multiplier is zero in this case because an increase in government spending is completely offset by a decrease in private investment.

2. (a) A Sequential Markets Equilibrium is:

(i) household's policy functions $c(k_{m,t-1}, k_{h,t-1}, K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$, $n(k_{m,t-1}, k_{h,t-1}, K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$, $i_m(k_{m,t-1}, k_{h,t-1}, K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$, $i_h(k_{m,t-1}, k_{h,t-1}, K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$;

(ii) firm's policy functions $N^d(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t}), K_m^d(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$;

(iii) prices $w(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t}), r(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$; and

(iv) law of motion of capital $K_{m,t} = g_m(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$ and $K_{h,t} = g_h(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t})$ such that

- Given prices and the law of motion of capital, household's policy functions solve the household's problem.

$$\max_{c_t, n_t, i_{m,t}, i_{h,t}} E \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

$$\begin{aligned} s.t. \quad & c_t + i_{m,t} + i_{h,t} \leq w_t n_t + r_t k_{m,t-1} \\ & h_t = z_{h,t} k_{h,t-1}^{1-\alpha_h} (X_{h,t}(1 - n_t))^{\alpha_h} \\ & k_{m,t} = (1 - \delta) k_{m,t-1} + i_{m,t} \\ & k_{h,t} = (1 - \delta) k_{h,t-1} + i_{h,t} \end{aligned}$$

- Given prices, firm's policy function solve the firm's problem.

$$\max_{K_{m,t}^d, N_t^d} [z_{m,t}(K_{m,t}^d)^{1-\alpha_m}(X_{m,t} N_t^d)^{\alpha_m} - r_t K_{m,t}^d - w_t N_t^d]$$

- Markets clear
  - $n_t = N_t = N_t^d$ (labor market)
  - $k_{m,t-1} = K_{m,t-1} = K_{m,t}^d$ (capital market)
  - $c_t + i_{m,t} + i_{h,t} = z_{m,t} K_{m,t-1}^{1-\alpha_m} (X_{m,t} N_t)^{\alpha_m}$ (goods market)
- Consistency of individual asset holdings and aggregate capital stock

$$\begin{aligned} & g_m(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t}) \\ & = (1 - \delta) K_{m,t-1} + i_m(K_{m,t-1}, K_{h,t-1}, K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t}) \\ & g_h(K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t}) \\ & = (1 - \delta) K_{h,t-1} + i_h(K_{m,t-1}, K_{h,t-1}, K_{m,t-1}, K_{h,t-1}, z_{m,t}, z_{h,t}) \end{aligned}$$

(b) **Note**: Defining RCE in an economy with growth is tricky. Let's assume $\gamma_m = \gamma_h = 1$ for the time being.

A Recursive Competitive Equilibrium is:

(i) household's value functions $V(k_m, k_h, K_m, K_h, z_m, z_h)$ and

policy functions $c(k_m, k_h, K_m, K_h, z_m, z_h), n(k_m, k_h, K_m, K_h, z_m, z_h)$, $i_m(k_m, k_h, K_m, K_h, z_m, z_h), i_h(k_m, k_h, K_m, K_h, z_m, z_h)$;

(ii) firm's policy functions $N^d(K_m, K_h, z_m, z_h), K^{m,d}(K_m, K_h, z_m, z_h)$;

(iii) prices $w(K_m, K_h, z_m, z_h), r(K_m, K_h, z_m, z_h)$; and

(iv) law of motion of capital $K'_m = g_m(K_m, K_h, z_m, z_h)$ and $K'_h = g_h(K_m, K_h, z_m, z_h)$ such that

- Given prices and the law of motion of capital, household's value and policy functions solve the household's Bellman equation.

$$V(k_m, k_h, K_m, K_h, z_m, z_h) = \max_{c,n,i_m,i_h} [u(c,h) + \beta E\{V(k'_m, k'_h, K'_m, K'_h, z'_m, z'_h)\}]$$

$$\begin{aligned} s.t. \quad & c + i_m + i_h \leq wn + rk \\ & h = z_h k_h^{1-\alpha_h}(1-n)^{\alpha_h} \\ & k'_m = (1-\delta)k_m + i_m \\ & k'_h = (1-\delta)k_h + i_h \end{aligned}$$

- Given prices, firm's policy function solve the firm's problem.

$$\max_{K_m^d, N^d} [z_m (K_m^d)^{1-\alpha_m}(N^d)^{\alpha_m} - rK_m^d - wN^d]$$

- Markets clear
  - $n = N = N^d$ (labor market)
  - $k_m = K_m = K_m^d$ (capital market)
  - $c + i_m + i_h = z_m K_m^{1-\alpha_m} N^{\alpha_m}$ (goods market)
- Consistency of individual asset holdings and aggregate capital stock

$$\begin{aligned} g_m(K_m, K_h, z_m, z_h) &= (1-\delta)K_m + i_m(K_m, K_h, K_m, K_h, z_m, z_h) \\ g_h(K_m, K_h, z_m, z_h) &= (1-\delta)K_h + i_h(K_m, K_h, K_m, K_h, z_m, z_h) \end{aligned}$$

(c) Formulate the lagrangian for the household's problem:

$$\begin{aligned} \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg[& \frac{c_t^{1-\gamma}}{1-\gamma} + \theta \frac{h_t^{1-\lambda}}{1-\lambda} \\ & + \lambda_t\{w_t n_t + r_t k_{m,t-1} - c_t - k_{m,t} + (1-\delta)k_{m,t-1} - k_{h,t} + (1-\delta)k_{h,t-1}\} \\ & + \mu_t\{z_{h,t} k_{h,t-1}^{1-\alpha_h}(X_{h,t}(1-n_t))^{\alpha_h} - h_t\} \Bigg] \end{aligned}$$

- FONC w.r.t. c_t:

$$\lambda_t = c_t^{-\gamma}$$

- FONC w.r.t. $k_{m,t}$:

$$\lambda_t = \beta E_t[\lambda_{t+1}(r_{t+1} + 1 - \delta)]$$

- FONC w.r.t. $k_{h,t}$:

$$\lambda_t = \beta E_t[\mu_{t+1}(1-\alpha_h)h_{t+1}/k_{h,t} + \lambda_{t+1}(1-\delta)]$$

- FONC w.r.t. h_t:

$$\mu_t = \theta h_t^{-\lambda}$$

- FONC w.r.t. n_t:

$$\lambda_t w_t = \mu_t \alpha_h h_t/(1-n_t)$$

and then impose market clearing conditions, firms' FONCs, add resource constraints etc.

(d) Denote the growth rate of $Y_t, K_{m,t}, K_{h,t}, I_{m,t}, I_{h,t}, C_t, H_t$ as g. Then from the conditions for the labor market:

$$C_t^{-\gamma} z_{m,t}\alpha_m K_{m,t-1}^{1-\alpha_m} X_{m,t}^{\alpha_m} N_t^{\alpha_m-1} = \theta H_t^{-\gamma} z_{h,t}\alpha_h K_{h,t-1}^{1-\alpha_h} X_{h,t}^{\alpha_h}(1-N_t)^{\alpha_h-1}$$

Assuming $\gamma_m = \gamma_h = g$ and taking logs and first differences,

$$-\gamma g + (1-\alpha_m)g + \alpha_m g = -\lambda g + (1-\alpha_h)g + \alpha_h g$$

then we need $\gamma = \lambda$ for the labor market condition to hold along the balanced growth path.

(e) Plug in the home production into the utility function:

$$\frac{c_t^{1-\gamma}}{1-\gamma} + \theta\frac{(z_{h,t}X_{h,t}(1-n_t))^{1-\lambda}}{1-\lambda}$$

so setting $\tilde{\theta}_t = \theta(z_{h,t}X_{h,t})^{1-\lambda}$ and $h_t = 1-n_t$ gives the desired result.