Homework 1 Answer Key

Econ 205B, Winter 2017

1. (a) A Sequential Markets Equilibrium is:

   (i) household's policy functions $c(i_{t-1}, k_{t-1}, I_{t-1}, K_{t-1}, z_t)$, $h(i_{t-1}, k_{t-1}, I_{t-1}, K_{t-1}, z_t), i(i_{t-1}, k_{t-1}, I_{t-1}, K_{t-1}, z_t)$;

   (ii) firm's policy functions $H^d(I_{t-1}, K_{t-1}, z_t), K^d(I_{t-1}, K_{t-1}, z_t)$;

   (iii) prices $w(I_{t-1}, K_{t-1}, z_t), r(I_{t-1}, K_{t-1}, z_t)$; and

   (iv) law of motion of capital $K_t = g(I_{t-1}, K_{t-1}, z_t)$ such that

   - Given prices and the law of motion of capital, household's policy functions solve the household's problem.

$$\max_{c_t, h_t, i_t} E \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$
$$s.t. \quad c_t + i_t \le w_t h_t + r_t k_{t-1}$$
$$k_t = (1-\delta)k_{t-1} + \frac{1}{2} i_t + \frac{1}{2} i_{t-1}$$

   - Given prices, firm's policy function solve the firm's problem.

$$\max_{K_t^d, H_t^d} [z_t F(K_t^d, H_t^d) - r_t K_t^d - w_t H_t^d]$$

   - Markets clear
     - $h_t = H_t = H_t^d$ (labor market)
     - $k_{t-1} = K_{t-1} = K_t^d$ (capital market)
     - $c_t + i_t = z_t F(K_t^d, H_t^d)$ (goods market)
   - Consistency of individual asset holdings and aggregate capital stock

$$g(I_{t-1}, K_{t-1}, z_t) = (1-\delta)K_{t-1} + \frac{1}{2} i(I_{t-1}, K_{t-1}, I_{t-1}, K_{t-1}, z_t) + \frac{1}{2} I_{t-1}$$

   (b) A Recursive Competitive Equilibrium is:

   (i) household's value functions $V(i_{-1}, k, I_{-1}, K, z)$ and policy functions $c(i_{-1}, k, I_{-1}, K, z), h(i_{-1}, k, I_{-1}, K, z), i(i_{-1}, k, I_{-1}, K, z)$;

   (ii) firm's policy functions $H^d(I_{-1}, K, z), K^d(I_{-1}, K, z)$;

   (iii) prices $w(I_{-1}, K, z), r(I_{-1}, K, z)$; and

   (iv) law of motion of capital $K' = g(I_{-1}, K, z)$ such that

- Given prices and the law of motion of capital, household's value and policy functions solve the household's Bellman equation.

$$V(i_{-1}, k, I_{-1}, K, z) = \max_{c,h,i}[u(c,h) + \beta E\{V(i, k', I, K', z')\}]$$
$$s.t. \quad c + i \leq wh + rk$$
$$k' = (1-\delta)k + \frac{1}{2}i + \frac{1}{2}i_{-1}$$

- Given prices, firm's policy function solve the firm's problem.

$$\max_{K^d, H^d}[zF(K^d, H^d) - rK^d - wH^d]$$

- Markets clear
  - $h = H = H^d$ (labor market)
  - $k = K = K^d$ (capital market)
  - $c + i = zF(K_t^d, H_t^d)$ (goods market)
- Consistency of individual asset holdings and aggregate capital stock

$$g(I_{-1}, K, z) = (1-\delta)K + \frac{1}{2}i(I_{-1}, K, I_{-1}, K, z) + \frac{1}{2}I_{-1}$$

(c) Solving through the sequential markets methodology, we construct a Lagrangian for the Household's optimization problem:

$$\begin{aligned}\mathcal{L} = \max_{c_t, k_t, i_t, h_t} E \sum_{t=0}^{\infty} \beta^t [u(c_t, h_t) \\ + \lambda_t\{w_t h_t + r_t k_{t-1} - i_t - c_t\} \\ + \mu_t\{(1-\delta)k_{t-1} + \frac{1}{2}i_t + \frac{1}{2}i_{t-1} - k_t\}]\end{aligned}$$

Taking First Order Conditions with respect to the decision variables. First with respect to c_t :

$$\frac{\partial \mathcal{L}}{\partial c_t} \implies u_1(c_t, h_t) = \lambda_t$$

With respect to k_t :

$$\frac{\partial \mathcal{L}}{\partial k_t} \implies \mu_t = \beta E_t[\lambda_{t+1} r_{t+1} + \mu_{t+1}(1-\delta)]$$

With respect to i_t:

$$\frac{\partial \mathcal{L}}{\partial i_t} \implies \lambda_t = \frac{1}{2}\mu_t + \frac{1}{2}\beta E_t[\mu_{t+1}]$$

With respect to h_t :

$$\frac{\partial \mathcal{L}}{\partial h_t} \implies u_2(c_t, h_t) + \lambda_t w_t = 0$$

Combining the consumption and labor conditions, we derive the optimal rule for intratemporal substitution between consumption and leisure:

$$u_2(c_t, h_t) + u_1(c_t, h_t) w_t = 0$$

Solving the Firm's problem will pin down the wage and rental rate of capital:

$$\max_{K_t^d, H_t^d} [z_t F(K_t^d, H_t^d) - r_t K_t^d - w_t H_t^d]$$

The First Order Conditions yield:

$$\frac{\partial \pi}{\partial K_t^d} \implies r_t = F_1(K_t^d, H_t^d)$$
$$\frac{\partial \pi}{\partial H_t^d} \implies w_t = F_2(K_t^d, H_t^d)$$

Finally, impose market-clearing conditions to these equations and add resource constraints, etc. and we get the equilibrium conditions.

(d) Taking the capital accumulation equation and log-linearizing it gives us:

$$\begin{aligned}
&K_t = (1-\delta)K_{t-1} + \frac{1}{2}I_t + \frac{1}{2}I_{t-1} \\
&\bar{K}(1+\hat{K}_t) = (1-\delta)\bar{K}(1+\hat{K}_{t-1}) + \frac{1}{2}\bar{I}(1+\hat{I}_t) + \frac{1}{2}\bar{I}(1+\hat{I}_{t-1}) \\
&\bar{K} + \bar{K}\hat{K}_t = \bar{K} + \bar{K}\hat{K}_t - \delta(\bar{K} + \bar{K}\hat{K}_{t-1}) + \frac{1}{2}(\bar{I} + \bar{I}\hat{I}_t + \bar{I} + \bar{I}\hat{I}_{t-1}) \\
&\bar{K}\hat{K}_t = (1-\delta)\bar{K}\hat{K}_{t-1} + \frac{1}{2}\bar{I}\hat{I}_t + \frac{1}{2}\bar{I}\hat{I}_{t-1}
\end{aligned}$$

Comparing to the standard capital accumulation equation:

$$\bar{K}\hat{K}_t = (1-\delta)\bar{K}\hat{K}_{t-1} + \bar{I}\hat{I}_t$$

Due to the time-to-build component, lagged investment is now also included. The outcome of this is that it now an increase in capital stock due to an increase in investment materializes slower.

2. (a) A Sequential Markets Equilibrium is:

(i) household's policy functions $c(k_{t-1}, K_{t-1}, a_t, z_t), h(k_{t-1}, K_{t-1}, a_t, z_t),$

$i(k_{t-1}, K_{t-1}, a_t, z_t)$;

(ii) firm's policy functions $H^d(K_{t-1}, a_t, z_t), K^d(K_{t-1}, a_t, z_t)$;

(iii) prices $w(K_{t-1}, a_t, z_t), r(K_{t-1}, a_t, z_t)$; and

(iv) law of motion of capital $K_t = g(K_{t-1}, a_t, z_t)$ such that

- Given prices and the law of motion of capital, household's policy functions solve the household's problem.

$$\max_{c_t, h_t, i_t} E \sum_{t=0}^{\infty} \beta^t \left[\frac{(a_t c_t)^{1-\sigma}}{1-\sigma} - \varphi \frac{h_t^{1+\eta}}{1+\eta} \right]$$
$$s.t. \quad c_t + i_t \leq w_t h_t + r_t k_{t-1}$$
$$k_t = (1-\delta) k_{t-1} + i_t$$

- Given prices, firm's policy function solve the firm's problem.

$$\max_{K_t^d, H_t^d} [z_t F(K_t^d, H_t^d) - r_t K_t^d - w_t H_t^d]$$

- Markets clear
  - $h_t = H_t = H_t^d$ (labor market)
  - $k_{t-1} = K_{t-1} = K_t^d$ (capital market)
  - $c_t + i_t = z_t F(K_t^d, H_t^d)$ (goods market)
- Consistency of individual asset holdings and aggregate capital stock

$$g(K_{t-1}, a_t, z_t) = (1-\delta) K_{t-1} + i(K_{t-1}, K_{t-1}, a_t, z_t)$$

(b) A Recursive Competitive Equilibrium is:

(i) household's value functions $V(k, K, a, z)$ and

policy functions $c(k, K, a, z), h(k, K, a, z), i(k, K, a, z)$;

(ii) firm's policy functions $H^d(K, a, z), K^d(K, a, z)$;

(iii) prices $w(K, a, z), r(K, a, z)$; and

(iv) law of motion of capital $K' = g(K, a, z)$ such that

- Given prices and the law of motion of capital, household's value and policy functions solve the household's Bellman equation.

$$V(k, K, a, z) = \max_{c, h, k'} \left[\frac{(ac)^{1-\sigma}}{1-\sigma} - \varphi \frac{h^{1+\eta}}{1+\eta} + \beta E\{V(k', K', a', z')\} \right]$$
$$s.t. \quad c + i \leq wh + rk$$
$$k' = (1-\delta) k + i$$

- Given prices, firm's policy function solve the firm's problem.

$$\max_{K^d, H^d} [zF(K^d, H^d) - rK^d - wH^d]$$

- Markets clear
  - $h = H = H^d$ (labor market)
  - $k = K = K^d$ (capital market)
  - $c + i = zF(K_t^d, H_t^d)$ (goods market)
- Consistency of individual asset holdings and aggregate capital stock

$$g(K, a, z) = (1 - \delta)K + i(K, K, a, z)$$

(c) The Euler condition continues to relate the marginal utility of consumption at period t and $t+1$, but the marginal utility of consumption is now equal to $a_t^{1-\sigma} c_t^{-\sigma}$, so we get

$$a_t^{1-\sigma} c_t^{-\sigma} = \beta E_t [a_{t+1}^{1-\sigma} c_{t+1}^{-\sigma} (r_{t+1} + 1 - \delta)]$$

In a log-linearized form, this becomes

$$\begin{aligned} \hat{c}_t &= -\sigma^{-1} \beta \bar{r} E_t r_{t+1} + E_t \hat{c}_{t+1} - \sigma^{-1}(1 - \sigma)(E_t \hat{a}_{t+1} - \hat{a}_t) \\ &= -\sigma^{-1} \beta \bar{r} E_t r_{t+1} + E_t \hat{c}_{t+1} - \sigma^{-1}(1 - \sigma)(\rho_a - 1)\hat{a}_t \end{aligned}$$

A positive realization of a_t makes consumption today more valuable if $\sigma < 1$ (less valuable if $\sigma > 1$). If $\rho_a = 1$, then the effect of t on the marginal utility of consumption is the same today and tomorrow, so given future consumption, the impact on current consumption would be zero.

(d) The marginal rate of substitution equals the real wage condition is

$$\frac{\varphi h_t^\eta}{a_t^{1-\sigma} c_t^{-\sigma}} = w_t \Rightarrow \eta \hat{h}_t - (1 - \sigma)\hat{a}_t + \sigma \hat{c}_t = \hat{w}_t$$

Inserting the log-linearized Euler equation above and the expression for real wages, a positive a_t increase (reduces) labor supply if $\sigma < 1$ ($\sigma > 1$) because if makes consumption more (less) valuable.

(e) The answer depends on the sign of $1 - \sigma$. Assume this is positive. Then a positive a_t leads to a rise in consumption (assuming $\sigma < 1$) and a rise in output via increased labor supply. So $\hat{c}$ and $\hat{y}$ move in the same direction. On the other hand, h and w moves in the opposite direction.

3. Since capital is freely movable across sectors, the total capital stock K is still the only

endogenous state variable.

$$\begin{aligned} V(K, z_1, z_2) &= \max_{K', K_1, K_2, H_1, H_2} [u(z_1 K_1^{\alpha} H_1^{1-\alpha}, H_1 + H_2) + \beta E\{V(K',\ z_1',\ z_2')\}] \\ s.t. \quad K' &= z_2 K_2^{\gamma} H_2^{1-\gamma} \\ K &= K_1 + K_2 \end{aligned}$$

4. Now, K_1 and K_2 are separate endogenous state variables (since they are both predetermined):

$$\begin{aligned} V(K_1, K_2, z_1, z_2) &= \max_{K_1', K_2', H_1, H_2} [u(z_1 K_1^{\alpha} H_1^{1-\alpha}, H_1 + H_2) + \beta E\{V(K_1',\ K_2',\ z_1',\ z_2')\}] \\ s.t. \quad K_1' + K_2' &= z_2 K_2^{\gamma} H_2^{1-\gamma} \end{aligned}$$