Economics 205A
Fall 2016
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Problem Set 5 Answer Key

Alternative answer for question 1

My answer on the answer key assumed that all capital income was taxed, as in the readings in Ljungqvist and Sargent. In class this year, I always assumed that capital income net of depreciation was taxed, as in the readings in Wickens. This provides the answer to Question 1 using the Wickens assumption.

You can compare the two. The post-tax net return to capital equals

$$\left(1-\tau_t^k\right)\left(f'\left(k_t\right)-\delta\right)$$

when depreciation is deducted, and equals

$$\left(1-\tau_t^k\right)f'\left(k_t\right)-\delta$$

when there is no depreciation allowance (ie. depreciation cannot be deducted before the tax is applied).

1. Consider the optimal capital income tax problem for the case in which household utility is given by

$$U_0=\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\sigma}}{1-\sigma}+\log\left(1-n_t\right)\right)$$

The production function is the standard constant returns to scale one, $y=f\left(k\right)$.

a) Set up the optimization problem for the household with proportional taxes on capital and labor income. Write down the first-order conditions for the household optimum.

The household solves

$$\max_{\{c_t,n_t,k_t\}}\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\sigma}}{1-\sigma}+\log\left(1-n_t\right)\right)$$

subject to

$$k_{t+1}+b_{t+1}=\left(1+r_t\right)b_t+\left(1+\left(1-\tau_t^k\right)\left(\nu_t-\delta\right)\right)k_t+\left(1-\tau_t^n\right)w_tn_t-c_t$$

and the solvency constraint

$$\lim_{t\to\infty}\left(\prod_{s+1}^{t}\frac{1}{1+r_s}\right)\left(b_{t+1}+k_{t+1}\right)\geq 0$$

for given initial k_0+b_0.

The first-order conditions for an interior optimum are

$$\frac{1}{1-n_t}=\left(1-\tau_t^n\right)w_tc_t^{-\sigma}$$
$$c_t^{-\sigma}=\left(1+\left(1-\tau_t^k\right)\left(\nu_t-\delta\right)\right)\beta c_{t+1}^{-\sigma}$$
$$c_t^{-\sigma}=\left(1+r_{t+1}\right)\beta c_{t+1}^{-\sigma}$$

which imply that $r_{t+1} = \left(1 - \tau_{t+1}^k\right)\left(\nu_{t+1}^k - \delta\right)$. The transversality conditions are

$$\lim_{t\to\infty} \beta^t c_t^{-\sigma} b_{t+1} = 0 \quad \text{and} \quad \lim_{t\to\infty} \beta^t c_t^{-\sigma} k_{t+1} = 0.$$

Let's add the factor market equilibrium conditions:

$$\nu_{t+1} = f_k\left(k_t, n_t\right) \quad \text{and} \quad w_t = f_n\left(k_t, n_t\right)$$

where I interpreted the production function as per capita instead of per unit of labor so it is the same as the class notes.

b) Set up the optimal tax problem for the government.

The planner solves

$$\max_{\{c_t, n_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \log\left(1 - n_t\right)\right)$$

subject to

$$k_{t+1} + b_{t+1} = \left(1 + r_t\right) b_t + \left(1 - \left(1 - \tau_t^k\right)\left(\nu_t - \delta\right)\right) k_t + \left(1 - \tau_t^n\right) w_t n_t - c_t$$
$$\frac{1}{1 - n_t} = \left(1 - \tau_t^n\right) w_t c_t^{-\sigma}$$
$$c_t^{-\sigma} = \left(1 - \left(1 - \tau_t^k\right)\left(\nu_t - \delta\right)\right) \beta c_{t+1}^{-\sigma}$$
$$\nu_{t+1} = f_k\left(k_t, n_t\right) \quad \text{and} \quad w_t = f_n\left(k_t, n_t\right)$$

and the resource identity,

$$k_{t+1} = \left(1 - \delta\right) k_t + f\left(k_t, n_t\right) - g_t - c_t,$$

given the constraints $k_{t+1} \geq 0$ and $\lim_{t\to\infty}\left(\prod_{s+1}^{t} \frac{1}{1+r_s}\right) b_{t+1} \leq 0$.

For the primal approach, we write the implementability condition as

$$c_0^{-\sigma}\left(1 + \left(1 - \tau_0^k\right)\left(\nu_0 - \delta\right)\right)\left(k_0 + b_0\right) + \sum_{t=0}^{\infty} \beta^t \left(\frac{n_t}{1 - n_t} - c_t^{1-\sigma}\right) = 0.$$

The planner's problem is

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \log\left(1 - n_t\right)\right)$$

subject to

$$\sum_{t=0}^{\infty} \beta^t \left(c_t^{1-\sigma} - \frac{n_t}{1 - n_t}\right) - c_0^{-\sigma}\left(1 + r_0\right)\left(k_0 + b_0\right) = 0,$$

and

$$k_{t+1} = \left(1 - \delta\right) k_t + f\left(k_t, n_t\right) - g_t - c_t$$

for all $\dot{t} > 0$. We assume that τ_0^k is constrained to be zero.

c) Derive the first-order conditions for the government's optimal policy.

The first-order conditions for the government's optimal policy are

$$\frac{\partial L}{\partial c_t} = \beta^t\left(c_t^{-\sigma} - \gamma_t + \mu\left(1 - \sigma\right) c_t^{-\sigma}\right) = 0 \quad \text{for } t \geq 1,$$

$$\frac{\partial L}{\partial c_0} = c_0^{-\sigma} - \gamma_0 + \mu\left(1 - \sigma\right) c_0^{-\sigma} + \mu\left(1 + r_0\right)\left(k_0 + b_0\right) \sigma c_0^{-\sigma-1} = 0 \quad \text{for } t = 0$$

$$\frac{\partial L}{\partial n_t} = \beta^t \left(\frac{-1}{1-n_t} + \gamma_t f_n (k_t, n_t) - \mu \frac{1}{(1-n_t)^2} \right) = 0 \quad \text{for } t \geq 0,$$

and

$$\frac{\partial L}{\partial k_{t+1}} = \beta^{t+1} \gamma_{t+1} (1 - \delta + f_k (k_{t+1}, n_{t+1})) - \beta^t \gamma_t = 0 \quad \text{for } t \geq 0.$$

d) Show (or explain) that the optimal capital income tax will equal zero for all $t > 1$. That is, the only positive tax on capital income for these preferences can be τ_1^k if we rule out a capital levy at time 0 (that is, $\tau_0^k = 0$ is fixed).

The first-order condition in c_t requires that

$$(1 + \mu (1 - \sigma)) c_t^{-\sigma} = \gamma_t.$$

Substituting this into the Euler condition,

$$\gamma_t = \beta \gamma_{t+1} (1 - \delta + f_k (k_{t+1}, n_{t+1})),$$

leads to

$$c_t^{-\sigma} = (1 - \delta + f_k (k_{t+1}, n_{t+1})) \beta c_{t+1}^{-\sigma}$$

which holds for $t \geq 1$. For $t = 0$, the Euler condition is

$$c_0^{-\sigma} = \left(\frac{1 + \mu (1 - \sigma)}{1 + \mu (1 - \sigma) + \mu \sigma c_0^{-1} (1 + r_0) (k_0 + b_0)} \right) (1 - \delta + f_k (k_1, n_1)) \beta c_1^{-\sigma}.$$

The household Euler condition,

$$c_t^{-\sigma} = \left(1 + \left(1 - \tau_{t+1}^k \right) \left(\nu_{t+1}^k - \delta \right) \right) \beta c_{t+1}^{-\sigma},$$

also holds. The two Euler conditions hold if

$$1 + \left(1 - \tau_{t+1}^k \right) \left(\nu_{t+1}^k - \delta \right) = 1 - \delta + f_k (k_{t+1}, n_{t+1})$$

which requires that $\tau_{t+1}^k = 0$ for all $t \geq 0$. The two Euler conditions for $t = 0$ imply

$$1 + \left(1 - \tau_1^k \right) \left(\nu_1^k - \delta \right) = \left(\frac{1 + \mu (1 - \sigma)}{1 + \mu (1 - \sigma) + \mu \sigma c_0^{-1} (1 + r_0) (k_0 + b_0)} \right) (1 - \delta + f_k (k_1, n_1))$$

which requires a positive τ_1^k if $\mu > 0$.