Economics 205A
Fall 2016
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Problem Set 4 Sample Answers

1. a) Ans: A competitive equilibrium is an allocation, $\{c_t, g_t, k_{t+1}\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that

(i) the household solves

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the budget identity,

$$a_{t+1} = (1 + r_t) a_t + w_t - T_t - c_t,$$

and the solvency constraint,

$$\lim_{t\to\infty} R_t^{-1} a_{t+1} \geq 0,$$

given initial assets a_0. The market discount factor is denoted $R_t^{-1} \equiv \prod_{s=1}^{t} \left(\frac{1}{1+r_s}\right)$ where R_0^{-1} is understood to equal 1.

(ii) each firm maximizes its profit for each $t \geq 0$,

$$\max_{n_t^d, k_t^d} \left(n_t^d f\left(\frac{k_t^d}{n_t^d}\right) - w_t n_t^d - \upsilon_t k_t^d \right)$$

(iii) goods, factor, and asset markets clear, taking taking the fiscal policy, $\{g_t, T_t, b_{t+1}\}_{t=0}^{\infty}$, as given. The market clearing conditions are

$$k_{t+1} = f(k_t) + (1 - \delta) k_t - g_t - c_t$$

$$n_t^d = n_t^s = 1, \qquad k_t^d = k_t,$$

$$a_t = k_t + b_t$$

and the government's budget constraint must be satisfied. The government's budget constraint is determined using the budget identity,

$$b_{t+1} = (1 + r_t) b_t + g_t - T_t$$

initial government debt, b_0, and government solvency,

$$\lim_{t\to\infty} R_t^{-1} b_{t+1} \leq 0.$$

b) Ans: First, derive the necessary conditions for the household:

$$u'(c_t) = \beta (1 + r_{t+1}) u'(c_{t+1}),$$

$$a_{t+1} = (1 + r_t) a_t + w_t - T_t - c_t,$$

and the solvency constraint,

$$\lim_{t\to\infty} \beta^t u'(c_t) a_{t+1} = 0 \quad \text{and} \quad \lim_{t\to\infty} R_t^{-1} a_{t+1} \geq 0,$$

given initial assets a_0.

Next, the first-order conditions for the firm and factor market equilibrium conditions give:

$$f(k_t) - k_t f'(k_t) = w_t \quad \text{and} \quad f'(k_t) = r_t + \delta.$$

The government budget identity, household budget identity, and asset market clearing condition lead to

$$\begin{aligned} a_{t+1} - b_{t+1} &= [(1+r_t)\,a_t + w_t - T_t - c_t] - [(1+r_t)\,b_t + g_t - T_t] \\ k_{t+1} &= (1+r_t)\,k_t + w_t - g_t - c_t. \end{aligned}$$

Using the factor market equilibrium conditions, we get the resource identity from taking the difference between the household and government budget identities,

$$k_{t+1} = f(k_t) + (1-\delta)\,k_t - g_t - c_t.$$

The conditions that determine the private sector allocation, $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and competitive equilibrium prices $\{w_t, r_t\}_{t=0}^{\infty}$ are the Euler condition, factor market equilibrium conditions, resource identity, and the transversality condition, $\lim_{t\to\infty} \beta^t u'(c_t)\,(k_{t+1} + b_{t+1}) = 0$. None of these depends on taxes, $\{T_t\}_{t=0}^{\infty}$, but the equilibrium depends on the sequence of expenditures $\{g_t\}_{t=0}^{\infty}$.

The path for b_{t+1} does depend on $\{T_t\}_{t=0}^{\infty}$ (as well as $\{g_t\}_{t=0}^{\infty}$ and b_0) using the government budget identity,

$$b_{t+1} = (1+r_t)\,b_t + g_t - T_t.$$

c) Ans: Just integrate the government budget identity imposing the solvencey condtion for the government to get

$$(1+r_0)\,b_0 = \sum_{t=0}^{\infty} R_t^{-1}\,(T_t - g_t) + \lim_{t\to\infty} R_t^{-1} b_{t+1}.$$

2. a) Ans: Let's begin with the representative firm. The firm will maximize profit given by

$$\pi_t = A k_t^{\alpha} g_t^{1-\alpha} - \upsilon_t k_t$$

leading to the first-order condition, $\alpha A k_t^{\alpha-1} g_t^{1-\alpha} = \upsilon_t$. Asset market equilibrium will tell us that $\upsilon_t = r_t + \delta$. Equilibrium profit is $\pi_t = (1-\alpha)\,A k_t^{\alpha} g_t^{1-\alpha}$.

We observe that there is no labor income but households own firms and profits are positive in equilibrium. This means that we write the household budget identity as

$$a_{t+1} - a_t = r_t a_t + \pi_t - T_t - c_t.$$

The representative household's optimization problem in decentralized equilibrium is

$$\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$a_{t+1} - a_t = r_t a_t + \pi_t - T_t - c_t,$$

$$\lim_{t\to\infty} R_t^{-1} a_{t+1} \geq 0$$

given initial assets a_0. The Euler condition and transversality condition are

$$c_t^{-\sigma} = (1 + r_{t+1})\,\beta c_{t+1}^{-\sigma}$$

and

$$\lim_{t\to\infty} \beta^t c_t^{-\sigma} a_{t+1} = 0 \qquad \Rightarrow \qquad \lim_{t\to\infty} R_t^{-1} a_{t+1} = 0.$$

Taxes are lump sum, so we know Ricardian equivalence holds. Equilibrium is given by the conditions:

$$\frac{c_{t+1}}{c_t} = \left(\left(1 - \delta + \alpha A \left(\frac{g_t}{k_t}\right)^{1-\alpha}\right)\beta\right)^{\frac{1}{\sigma}},$$

$$k_{t+1} - k_t = A k_t^{\alpha} g_t^{1-\alpha} - \delta k_t - g_t - c_t,$$

and

$$\lim_{t\to\infty} \beta^t c_t^{-\sigma} k_{t+1} = 0$$

given k_0 and $k_{t+1} \geq 0$.

b) Ans: Let $\frac{g_t}{k_t} = \frac{g_0}{k_0} \equiv B > 0$. The marginal product of capital is

$$\alpha A k_t^{\alpha-1} g_t^{1-\alpha} = \alpha A B^{1-\alpha},$$

and the Euler condition is

$$\frac{c_{t+1}}{c_t} = \left(1 - \delta + \alpha A B^{1-\alpha}\right)^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}.$$

The resource identity is

$$\begin{aligned} k_{t+1} - k_t &= A B^{1-\alpha} k_t - \delta k_t - B k_t - c_t \\ &= \left(A B^{1-\alpha} - \delta - B\right) k_t - c_t. \end{aligned}$$

The balanced growth rate equals $\left(\left(1 - \delta + \alpha A B^{1-\alpha}\right)\beta\right)^{\frac{1}{\sigma}} - 1$, and the solution for consumption is found by integrating the resource identity using the Euler condition. This is

$$c_t = \left[\left(1 - \delta + A B^{1-\alpha} - B\right) - \left(1 - \delta + \alpha A B^{1-\alpha}\right)^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}\right] k_t.$$

c) Ans: To find the command optimum we solve

$$\max_{\{c_t, g_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma}$$

subject to

$$k_{t+1} - k_t = A k_t^{\alpha} g_t^{1-\alpha} - \delta k_t - g_t - c_t$$

and the inequalities, $k_{t+1} \geq 0$ and $g_t \geq 0$.

The necessary conditions are the Euler condition,

$$\frac{c_{t+1}}{c_t} = \left(1 - \delta + \alpha A \left(\frac{g_t}{k_t}\right)^{1-\alpha}\right)^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}},$$

the first-order condition for g_t,

$$(1 - \alpha)\, A \left(\frac{g_t}{k_t}\right)^{-\alpha} = 1,$$

the resource identity,

$$\frac{k_{t+1}-k_t}{k_t}=A\left(\frac{g_t}{k_t}\right)^{1-\alpha}-\delta-\frac{g_t}{k_t}-\frac{c_t}{k_t},$$

the transversality condition,

$$\lim_{t\to\infty}\beta^t c_t^{-\sigma}k_{t+1}=0,$$

and the initial condition, k_0.

Using these conditions, we have that

$$(1-\alpha)\,A\left(\frac{g_t}{k_t}\right)^{-\alpha}=1\qquad\Rightarrow\qquad\frac{g_t}{k_t}=(1-\alpha)^{\frac{1}{\alpha}}A^{\frac{1}{\alpha}}$$

so that

$$\begin{aligned}\frac{c_{t+1}-c_t}{c_t} &= \left(1-\delta+\alpha A\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1-\alpha}{\alpha}}\right)^{\frac{1}{\sigma}}\beta^{\frac{1}{\sigma}}-1\\ &= \left(1-\delta+\alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}\right)^{\frac{1}{\sigma}}\beta^{\frac{1}{\sigma}}-1\end{aligned}$$

and

$$\begin{aligned}\frac{k_{t+1}-k_t}{k_t} &= (1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}-(1-\alpha)^{\frac{1}{\alpha}}A^{\frac{1}{\alpha}}-\delta-\frac{c_t}{k_t}\\ &= \alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}-\delta-\frac{c_t}{k_t}.\end{aligned}$$

The command optimum is given by the growth rate,

$$\frac{k_{t+1}-k_t}{k_t}=\frac{c_{t+1}-c_t}{c_t}=\left(1-\delta+\alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}\right)^{\frac{1}{\sigma}}\beta^{\frac{1}{\sigma}}-1$$

and the consumption function,

$$c_t=\left[\alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}+(1-\delta)-\left(1-\delta+\alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}\right)^{\frac{1}{\sigma}}\beta^{\frac{1}{\sigma}}\right]k_t.$$

This solution provides the optimal value of the parameter B as $(1-\alpha)^{\frac{1}{\alpha}}A^{\frac{1}{\alpha}}$, and demonstrates that the optimal g_t is a constant proportion of k_t. For the case in which $\rho=r_t$ for the optimal

$$r_t=\alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}-\delta,$$

the growth rate is zero and the solution for consumption is

$$\begin{aligned}c_t &= \left[\alpha\,(1-\alpha)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}-\delta\right]k_t\\ &= \rho k_t.\end{aligned}$$

d) Ans: The production function displays constant returns to scale, so that

$$F(k,g)=\frac{\partial F}{\partial k}k+\frac{\partial F}{\partial g}g.$$

In equilibrium for the firm, profit is

$$\pi=F(k,g)-\upsilon k=F(k,g)-\frac{\partial F}{\partial k}k=\frac{\partial F}{\partial g}g\quad\text{for}\quad\upsilon=r+\delta$$

That is, the firm's profit is the contribution of g to output which it does not pay for. If the firm paid for its input of productive public expenditures, the price of g would be

$$\tau_t = \frac{\partial F}{\partial g_t} = (1-\alpha) A k_t^{\alpha} g_t^{-\alpha}, \quad \text{and} \quad \upsilon_t k_t + \tau_t g_t = F(k_t, g_t).$$

If the firm faced a market for both k and g, the equilibrium prices would be $\upsilon_t = \alpha A k_t^{\alpha-1} g_t^{1-\alpha}$ and $\tau_t = (1-\alpha) A k_t^{\alpha} g_t^{-\alpha}$. The profit for public enterprise would be $\pi_t^g = g_t - \tau_t g_t$. When this profit is maximized, $\tau_t = 1$ and the competitive equilibrium allocation is the Pareto optimum found in part c.

The lump-sum tax for a balanced budget policy is $T_t = g_t$. In decentralized equilibrium, profits are $(1-\alpha) F(k,g) = (1-\alpha) A k_t^{\alpha} g_t^{1-\alpha}$. For a ratio, g_t/k_t less than the optimal ratio of $(1-\alpha)^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}}$, profits exceed the cost of public spending. The lump-sum tax imposed is less than 100% of profits. In the optimum, for $\frac{\partial F}{\partial g} = 1$, the lump-sum tax equals a 100% profit tax. This observation elaborates the interpretation of firm profits in the decentralized equilibrium: these are the social value of expenditures g_t. The social cost of government inputs in the amount g_t is just g_t. In the optimum, the marginal cost of g will equal the marginal benefit of g. Under constant returns to scale in production of goods (ie. $F(k,g)$), the exact cost of producing public goods equals their marginal benefit times the amount produced in the optimum.

3. a) Ans: The household solves the optimization problem:

$$\max_{\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

subject to

$$a_{t+1} = (1+r_t) a_t + (1-\tau_t) w_t n_t - c_t,$$
$$1 \geq n_t \geq 0$$

and

$$\lim_{T\to\infty} R_T^{-1} a_{T+1} \geq 0$$

given a_0. The necessary conditions (for an interior optimum) are

$$u_c(c_t, 1-n_t) = (1+r_{t+1}) \beta u_c(c_{t+1}, 1-n_{t+1}),$$
$$u_\ell(c_t, 1-n_t) = (1-\tau_t) w_t u_c(c_t, 1-n_t),$$
$$a_{t+1} = (1+r_t) a_t + (1-\tau_t) w_t n_t - c_t,$$
$$\lim_{T\to\infty} \beta^T u_c(c_T, 1-n_T) a_{T+1} = 0 \quad \text{and} \quad \lim_{T\to\infty} R_T^{-1} a_{T+1} \geq 0.$$

b) Ans: The government budget identity is

$$b_{t+1} = (1+r_t) b_t - \tau_t w_t n_t + g_t$$

and the government solvency constraint is

$$\lim_{T\to\infty} R_T^{-1} b_{T+1} \leq 0.$$

c) Ans: Simply introduce a firm that rents capital and hires labor supplied by households to produce output according to a constant returns to scale production function, $y_t = n_t f\left(\frac{k_t}{n_t}\right)$.

A decentralized equilibrium is an allocation $\{(c_t, n_t, k_t, g_t)\}_{t=0}^{\infty}$, prices $\{(w_t, r_t)\}_{t=0}^{\infty}$, and a government policy $\{(g_t, \tau_t, b_t)\}$ such that

a) Households maximize utility over their budget sets given prices and government policy, and firms maximize profit given prices and government policy;

b) The allocation is feasible (that is, goods and factor markets clear); and

c) The government policy satisfies the government's budget constraint given the allocation and prices.

What does this mean? Households and firms take government expenditures and tax rates as given. The government must choose its tax rates so that it remains solvent given equilibrium prices and its expenditures, $\{g_t\}$.

The conditions are the necessary conditions for the household in part c,

$$u_c\left(c_t, 1-n_t\right) = \left(1+r_{t+1}\right)\beta u_c\left(c_{t+1}, 1-n_{t+1}\right),$$

$$u_\ell\left(c_t, 1-n_t\right) = \left(1-\tau_t\right) w_t u_c\left(c_t, 1-n_t\right),$$

$$a_{t+1} = \left(1+r_t\right) a_t + \left(1-\tau_t\right) w_t n_t - c_t$$

and

$$\lim_{T\to\infty} \beta^T u_c\left(c_T, 1-n_T\right) a_{T+1} = 0;$$

the factor market equilibrium conditions (from firm profit maximization),

$$r_t = f'\left(\frac{k_t}{n_t}\right) - \delta \quad \text{and} \quad w_t = f\left(\frac{k_t}{n_t}\right) - \frac{k_t}{n_t} f'\left(\frac{k_t}{n_t}\right);$$

the budget identity and solvency constraint for the government,

$$b_{t+1} = \left(1+r_t\right) b_t - \tau_t w_t n_t + g_t \quad \text{and} \quad \lim_{T\to\infty} R_T^{-1} b_{T+1} \le 0;$$

and the market clearing conditions,

$$k_{t+1} - k_t = f\left(k_t\right) - \delta k_t - g_t - c_t$$

and

$$a_t = k_t + b_t.$$

The inequality constraint $k_t \ge 0$ for all $t > 0$ must be satisfied, as well.

The household transversality condition assures that the government solvency constraint holds, while the household solvency constraint and restriction that k is non-negative imply that the government solvency constraint binds (holds with equality). We also note that the asset market equilibrium condition reduces the set of three identities, household budget identity, government budget identity, and resource identity, to two conditions. One is redundant in equilibrium.

4. a) Ans: The necessary conditions are the Euler condition,

$$c_t^{-\sigma} = \beta\left(1+r\right) c_{t+1}^{-\sigma} = c_{t+1}^{-\sigma}$$

the first-order condition between the consumption and labor supply (you can call this the labor-leisure choice),

$$c_t^{-\sigma} = \frac{1}{(1-\tau_t)\, w_t} n_t^{\gamma},$$

the household budget identity,

$$a_{t+1} = (1+r)\, a_t + (1-\tau_t)\, w_t n_t - c_t,$$

the transversality condition,

$$\lim_{t\to\infty} \beta^t c_t^{-\sigma} a_{t+1} = 0,$$

and the solvency condition, $\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t a_{t+1} \geq 0$, plus the initial condition, a_0. Using the Euler condition, the transversality condition implies that the solvency constraint holds with equality, $\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t a_{t+1} = 0$.

b) Ans: Begin with the government budget identity and solvency condition to form the government's budget constraint,

$$(1+r)\, b_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (g - \tau w_t n_t) \leq 0,$$

for constant g_t, τ_t and r_t. Using the household transversality condition, this will hold with equality so that

$$(1+r)\, b_0 + \frac{1+r}{r} g = \tau \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t n_t.$$

Next, write out the budget constraint for the household as

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t \leq (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (1-\tau)\, w_t n_t,$$

and impose equality by the transversality condition. Using the Euler condition to simplify, we have

$$c_0 = r a_0 + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (1-\tau)\, w_t n_t.$$

Substituting the budget constraint for the government into the household budget constraint, we get

$$\begin{aligned} c_0 &= r a_0 + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t n_t - \frac{r}{1+r} \tau \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t n_t \\ &= r\,(a_0 - b_0) + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t n_t - g \\ &= r\,(a_0 - b_0) + \widetilde{w_0 n_0} - g. \end{aligned}$$

where permanent labor income is

$$\widetilde{w_0 n_0} \equiv \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t n_t.$$

c) Ans: Denote consumption and labor supply for this equilibrium by $\widehat{c}_t$ and $\widehat{n}_t$ to distinguish these from their values in part b. Because consumption is constant over time for $r = \rho$, labor supply changes at time $T+1$ and relative to the answer in part b for any given sequence of wage rates, w_t. Using the first-order condition,

$$c_t^{-\sigma} = \frac{1}{(1-\tau_t)\, w_t} n_t^{\gamma},$$

we find that

$$\widehat{n}_T = \left(\frac{w_T}{(1-\tau')\, w_{T+1}} \right)^{\frac{1}{\gamma}} \widehat{n}_{T+1}$$

and

$$\widehat{n}_T^{\gamma} = w_T \widehat{c_0}^{-\sigma}.$$

We can see that labor supply falls for a given wage rate with the imposition of the tax. We need to find out how c changes to see how labor supply changes before the tax.

Again, we start with the government budget constraint (holding with equality),

$$(1+r)\, b_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t g - \sum_{t=T+1}^{\infty} \left(\frac{1}{1+r} \right)^t \tau' w_t \widehat{n}_t = 0$$

which can be rearranged as

$$r b_0 + g = \frac{r}{1+r} \sum_{t=T+1}^{\infty} \left(\frac{1}{1+r} \right)^t \tau' w_t \widehat{n}_t$$

The household budget constraint is

$$\begin{aligned} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \widehat{c}_t &= (1+r)\, a_0 + \sum_{t=0}^{T} \left(\frac{1}{1+r} \right)^t w_t \widehat{n}_t + \sum_{t=T+1}^{\infty} \left(\frac{1}{1+r} \right)^t (1-\tau')\, w_t \widehat{n}_t \\ &= (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t w_t \widehat{n}_t - \sum_{t=T+1}^{\infty} \left(\frac{1}{1+r} \right)^t \tau' w_t \widehat{n}_t \end{aligned}$$

Using the Euler condition, we have

$$\widehat{c}_0 = r a_0 + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t w_t \widehat{n}_t - \frac{r}{1+r} \sum_{t=T+1}^{\infty} \left(\frac{1}{1+r} \right)^t \tau' w_t \widehat{n}_t,$$

and substituting the government budget constraint into this expression for $\widehat{c}_0$ leads to

$$\widehat{c}_0 = r\,(a_0 - b_0) + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t w_t \widehat{n}_t - g.$$

Consumption, $\widehat{c}_0$, will differ from c_0 because permanent labor income $\frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t w_t \widehat{n}_t$ changes with tax rates for every $t \geq 0$. That is,

$$c_0 - \widehat{c}_0 = \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t w_t \left(n_t - \widehat{n}_t \right).$$

d) Ans: The timing of taxes matters with elastic labor supply and proportionate taxes on labor earnings because labor supply changes whenever tax rates change. For a given stream of public expenditures, changing the timing of taxes changes tax rates over time and changes labor supply over time. In the example, interest rates and wages are exogenous. Even in this case, labor supply changes when the timing of tax revenue collection is altered. This changes permanent income through permanent labor income changing consumption.

Not expected analysis:

Let $w_t = w$ be constant over time to show with the model. The change in labor supply at time $T+1$is given by

$$\widehat{n}_{T+1} = \left(1-\tau'\right)^{\frac{1}{\gamma}} \widehat{n}_T,$$

and at time $t=0$ by

$$\frac{\widehat{n}_0^{\gamma}}{n_0^{\gamma}} = \left(\frac{1}{1-\tau}\right) \frac{\widehat{c}_0^{-\sigma}}{c_0^{-\sigma}}$$

so that the difference in consumption is given by

$$c_0 - \widehat{c}_0 = \frac{r}{1+r}\left[\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t w n_0 \left(1-\left(\frac{1}{1-\tau}\right)^{\frac{1}{\gamma}}\left(\frac{\widehat{c}_0}{c_0}\right)^{\frac{-\sigma}{\gamma}}\right) + \sum_{t=T+1}^{\infty}\left(\frac{1}{1+r}\right)^t w n_0 \left(1-\left(\frac{1-\tau'}{1-\tau}\right)^{\frac{1}{\gamma}}\left(\frac{\widehat{c}_0}{c_0}\right)^{\frac{-\sigma}{\gamma}}\right)\right]$$

where

$$n_0 = (1-\tau)^{\frac{1}{\gamma}} w^{\frac{1}{\gamma}} c_0^{\frac{-\sigma}{\gamma}}.$$

Putting these together, we have

$$\begin{aligned}
c_0 - \widehat{c}_0 &= \frac{r}{1+r} w^{1+\frac{1}{\gamma}} \left[\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t \left((1-\tau)^{\frac{1}{\gamma}} c_0^{\frac{-\sigma}{\gamma}} - \widehat{c}_0^{\frac{-\sigma}{\gamma}}\right) + \sum_{t=T+1}^{\infty}\left(\frac{1}{1+r}\right)^t \left((1-\tau)^{\frac{1}{\gamma}} c_0^{\frac{-\sigma}{\gamma}} - \left(1-\tau'\right)^{\frac{1}{\gamma}} \widehat{c}_0^{\frac{-\sigma}{\gamma}}\right)\right] \\
&= w^{1+\frac{1}{\gamma}} \left[\left(1-\left(\frac{1}{1+r}\right)^{T+1}\right)\left((1-\tau)^{\frac{1}{\gamma}} c_0^{\frac{-\sigma}{\gamma}} - \widehat{c}_0^{\frac{-\sigma}{\gamma}}\right) + \left(\frac{1}{1+r}\right)^{T+1}\left((1-\tau)^{\frac{1}{\gamma}} c_0^{\frac{-\sigma}{\gamma}} - \left(1-\tau'\right)^{\frac{1}{\gamma}} \widehat{c}_0^{\frac{-\sigma}{\gamma}}\right)\right] \\
&= \widehat{c}_0^{\frac{-\sigma}{\gamma}} w^{1+\frac{1}{\gamma}} \left[\left((1-\tau)^{\frac{1}{\gamma}}\left(\frac{c_0}{\widehat{c}_0}\right)^{\frac{-\sigma}{\gamma}} - 1\right) + \left(\frac{1}{1+r}\right)^{T+1}\left(1-\left(1-\tau'\right)^{\frac{1}{\gamma}}\right)\right]
\end{aligned}$$

We need to use the government budget constraint to try to sign this expression. The government budget tells us that

$$\frac{r}{1+r}\sum_{t=T+1}^{\infty}\left(\frac{1}{1+r}\right)^t \tau' w \widehat{n}_{T+1} = \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \tau w n_0.$$

With substitution for $\widehat{n}_{T+1}$ and simplification, this becomes

$$\left(\frac{1}{1+r}\right)^{T+1} \tau' \left(1-\tau'\right)^{\frac{1}{\gamma}} \widehat{n}_0 = \tau n_0$$

and then

$$\left(\frac{1}{1+r}\right)^{T+1} \frac{\tau'}{\tau} \left(1-\tau'\right)^{\frac{1}{\gamma}} = (1-\tau)^{\frac{1}{\gamma}} \left(\frac{c_0}{\widehat{c}_0}\right)^{\frac{-\sigma}{\gamma}}.$$

Substituting this last result into the difference in consumptions equation leads to

$$c_0 - \widehat{c}_0 = \widehat{c}_0^{\frac{-\sigma}{\gamma}} w^{1+\frac{1}{\gamma}} \left[\left(\frac{1}{1+r}\right)^{T+1}\left(\frac{\tau'}{\tau} - 1\right)\left(1-\tau'\right)^{\frac{1}{\gamma}} + \left(\frac{1}{1+r}\right)^{T+1} - 1\right]$$

which shows that the choice of T and the elastiticities $\sigma > 0$ and $\gamma > 0$ will determine whether consumption rises or falls as taxes are postponed. The last two conditions are needed to find $\widehat{c}_0$ and τ' given c_0 and τ which depend on g, b_0, and w, along with the parameters of utility.