Economics 205A

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Problem Set 3: Sample Answers

1. a) Ans: The household's optimization problem is

$$\max_{\{c_t, a_{t+1}\}_{t-0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

subject to the budget identity

$$a_{t+1} - a_t = r_t a_t + w_t - c_t$$

and the solvency constraint

$$\lim_{T\to\infty} \prod_{t=1}^{T} \left(\frac{1}{1+r_t}\right) a_{T+1} \geq 0,$$

given initial financial wealth a_0.

b) Ans: The necessary conditions for an optimum are the Euler condition

$$c_t^{-\sigma} = \beta \left(1 + r_{t+1}\right) c_{t+1}^{-\sigma},$$

the budget identity

$$a_{t+1} - a_t = r_t a_t + w_t - c_t,$$

the transversality condition,

$$\lim_{T\to\infty} \beta^T c_T^{-\sigma} a_{T+1} = 0,$$

and the solvency constraint, $\lim_{T\to\infty} \prod_{t=1}^{T} \left(\frac{1}{1+r_t}\right) a_{T+1} \geq 0$, and initial condition, a_0.

c) Ans: We integrate the Euler condition to obtain

$$c_0^{-\sigma} = \beta^t \prod_{s=1}^{t} \left(1 + r_s\right) c_t^{-\sigma}$$

for all c_t, $t \geq 1$. Using this to substitute for marginal utility of $c_t^{-\sigma}$ in the transversality condition leads to

$$\lim_{T\to\infty} \beta^T c_T^{-\sigma} a_{T+1} = \lim_{T\to\infty} \beta^T c_0^{-\sigma} \beta^{-T} \prod_{t=1}^{T} \left(\frac{1}{1+r_t}\right) a_{T+1} = c_0^{-\sigma} \lim_{T\to\infty} \prod_{t=1}^{T} \left(\frac{1}{1+r_t}\right) a_{T+1} = 0.$$

The transversality condition requires the solvency constraint to bind.

d) Ans: We begin by solving the budget identity forward to get the identity,

$$\sum_{t=0}^{\infty} \prod_{s=1}^{t} \left(\frac{1}{1+r_s}\right) c_t = \left(1 + r_0\right) a_0 + \sum_{t=0}^{\infty} \prod_{s=1}^{t} \left(\frac{1}{1+r_s}\right) w_t - \lim_{T\to\infty} \prod_{t=1}^{T} \left(\frac{1}{1+r_t}\right) a_{T+1},$$

where $\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)$ is defined to be 1 for $t=0$. Using the transversality condition,

$$\lim_{T\to\infty}\beta^T c_T^{-\sigma} a_{T+1} = c_0^{-\sigma}\lim_{T\to\infty}\prod_{t=1}^{T}\left(\frac{1}{1+r_t}\right)a_{T+1} = 0,$$

this identity becomes the intertemporal budget constraint,

$$\sum_{t=0}^{\infty}\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)c_t = (1+r_0)\,a_0 + \sum_{t=0}^{\infty}\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)w_t.$$

Using the Euler condition to substitute for c_t in terms of c_0 from part c, the constraint becomes

$$\sum_{t=0}^{\infty}\beta^{\frac{t}{\sigma}}\prod_{s=1}^{t}(1+r_s)^{\frac{1}{\sigma}}\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)c_0 = (1+r_0)\,a_0 + \sum_{t=0}^{\infty}\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)w_t.$$

This equation rearranges to get the consumption at time 0,

$$c_0 = \left[\sum_{t=0}^{\infty}\beta^{\frac{t}{\sigma}}\prod_{s=1}^{t}(1+r_s)^{\frac{1}{\sigma}}\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)\right]^{-1}\left[(1+r_0)\,a_0 + \sum_{t=0}^{\infty}\prod_{s=1}^{t}\left(\frac{1}{1+r_s}\right)w_t\right].$$

e) Ans: Letting $r_t = r$ constant, the last equation becomes

$$c_0 = \left(1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}\right)\left[(1+r)\,a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t w_t\right],$$

which can be rewritten

$$\begin{aligned} c_0 &= \frac{1+r}{r}\left(1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1-\sigma}{\sigma}}\right)(ra_0+\widetilde{w}_0) \\ &= \left(1+\frac{1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{r}\right)(ra_0+\widetilde{w}_0) \end{aligned}$$

where $\widetilde{w}_0 = \left(\frac{r}{1+r}\right)\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t w_t$. We can write this for any t as

$$c_t = \left(1+\frac{1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{r}\right)(ra_t+\widetilde{w}_t).$$

If $\rho > r$, the consumption function shows that $c_t > ra_t + \widetilde{w}_t$ because $1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}} > 0$. The Euler condition, $c_{t+1} = c_t\left(\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}\right)$, shows that $c_{t+1} < c_t$ for $\rho > r$. The budget identity,

$$\begin{aligned} a_{t+1}-a_t &= ra_t + w_t - c_t \\ &= (w_t - \widetilde{w}_t) - \left(\frac{1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{r}\right)(ra_t+\widetilde{w}_t) \end{aligned}$$

shows that savings is lower when $\rho > r$ than when $\rho = r$ (again, because $1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}} > 0$).

If $\rho < r$, $c_t < ra_t + \widetilde{w}_t$ because $1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}} < 0$. Consumption $c_{t+1} = c_t\left(\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}\right) > c_t$ in this case, and savings is higher when $\rho < r$ than when $\rho = r$.

2. a) Ans: Household optimization implies that $c_t = c_{t+1}$ for all $t \geq 0$ (because $\beta = \frac{1}{1+r}$, the Euler condition is $u'(c_s) = u'(c_{s+1})$). The budget identity, solvency constraint, transversality condition and initial condition together give us the intertemporal budget constraint,

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t = (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t.$$

Substituting $c_t = c_0$ for all $t \geq 0$ leads to the consumption function for $t = 0$,

$$c_0 = ra_0 + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t.$$

Substituting in the process for w_t, consumption is given by

$$\begin{aligned} c_t &= ra_0 + \frac{r}{1+r} \left(\varepsilon + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w\right) \\ &= ra_0 + w + \frac{r}{1+r}\varepsilon, \end{aligned}$$

for $w_0 = w + \varepsilon$ and $w_t = w$ for all $t > 0$.

For $w_t = w+\ \varepsilon$ for all$t \geq 0$, consumption is given by

$$\begin{aligned} c_t &= ra_0 + \frac{r}{1+r} \left(\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (w+\varepsilon)\right) \\ &= ra_0 + w + \varepsilon. \end{aligned}$$

In the first case, the extra income ε lasts only one period. It is a temporary income increase. The permanent income part of this income increase is $\frac{r}{1+r}\varepsilon$. Consumption permanently rises by the permanent income increase, $\frac{r}{1+r}\varepsilon$, while savings at $t = 0$ increases by the transitory part of the income increase, $\frac{1}{1+r}\varepsilon$. Permanent labor income at date 0 is $\widetilde{w}_0 = w + \frac{r}{1+r}\varepsilon$, and transitory income is $\frac{1}{1+r}\varepsilon$.

In the second case, the income increase is permanent. The entire amount, ε, is permanent income and consumption rises by this amount. The transitory part of the income increase in the second case (ε is permanent) is zero. Savings at time 0 does not rise. Permanent income in this case is $\widetilde{w}_0 = w+\varepsilon$. Reiterating, consumption rises one-for-one with permanent income and savings rises one-for-one with transitory income.

b) Ans: The process followed by w_t iterated forward gives

$$w_t = w + \theta^t \varepsilon.$$

Substituting this into the solution for consumption at time 0 gives us

$$\begin{aligned} c_0 &= ra_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t w_t \\ &= ra_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(w+\theta^t\varepsilon\right) \\ &= ra_0 + w + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{\theta}{1+r}\right)^t \varepsilon \\ &= ra_0 + w + \frac{r}{1+r-\theta}\varepsilon \end{aligned}$$

For $\theta = 0$, the income increase lasts only one period and consumption rises by permanent income, $\frac{r}{1+r}\varepsilon$. For $\theta = 1$, the increase is permanent and consumption rises by ε. For θ between 0 and 1, the income increase is persistent but not permanent. Consumption rises by the amount of permanent income, which is $ra_0 + w + \frac{r}{1+r-\theta}\varepsilon$ and increases by $\frac{r}{1+r-\theta}\varepsilon$ with ε. Savings at time 0 is given by

$$\begin{aligned} a_1 - a_0 &= ra_0 + w + \varepsilon - c_0 \\ &= \left(1 - \frac{r}{1+r-\theta}\right)\varepsilon = \frac{1-\theta}{1+r-\theta}\varepsilon. \end{aligned}$$

The quantity, $\frac{1-\theta}{1+r-\theta}\varepsilon$, is transitory income at $t = 0$ and $\frac{r}{1+r-\theta}\varepsilon$ is permanent income at $t = 0$ (note that these add to 1).

c) Ans: This is an anticipated permanent increase in earnings arriving at time T. Consumption at time t is given by

$$\begin{aligned} c_0 &= ra_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t w_t \\ &= ra_0 + \frac{r}{1+r}\sum_{t=0}^{T-1}\left(\frac{1}{1+r}\right)^t w + \frac{r}{1+r}\sum_{t=T}^{\infty}\left(\frac{1}{1+r}\right)^t (w+\varepsilon) \\ &= ra_0 + w + \left(\frac{1}{1+r}\right)^T \varepsilon. \end{aligned}$$

The increase in c_0 is given by the increase in permanent income, which equals the present value increase in labor income at time 0, $\left(\frac{1}{1+r}\right)^T \varepsilon$. Because labor income does not rise until $t = T$, household consumption exceeds household income and savings is negative. The household consumes out of positive savings (if a_0 is positive and large enough) or borrows (a_1 is negative) at time 0. Once time T is reached, labor income is constant and consumption equals permanent income. For all $t \geq T$ forwards, consumption is

$$c_T = ra_T + w + \varepsilon$$

and savings is zero, $a_t = a_T$ for all $t \geq T$. Since consumption is constant over time before and after T

(because $r = \rho$), we also have that

$$\begin{aligned} c_T &= ra_T + w + \varepsilon \\ &= c_0 \\ &= ra_0 + w + \left(\frac{1}{1+r}\right)^T \varepsilon. \end{aligned}$$

You may notice that

$$a_T = a_0 + \frac{1}{r}\left(\frac{1}{1+r}\right)^T \varepsilon - \frac{1}{r}\varepsilon < a_0.$$

3. a) Ans: The household's optimization problem is

$$\max_{\{c_t, \ell_t, n_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\log c_t + \frac{\ell_t^{1-\gamma}}{1-\gamma}\right)$$

subject to the budget identity

$$a_{t+1} - a_t = ra_t + w_t n_t - c_t,$$

the leisure endowment constraint,

$$n_t + \ell_t \leq 1,$$

the non-negativity constraints for labor supply and leisure consumption,

$$n_t \geq 0 \qquad \text{and} \qquad \ell_t \geq 0$$

and the solvency constraint

$$\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t a_{t+1} \geq 0,$$

given initial financial wealth a_0.

b) Ans: The necessary conditions for an optimum are the Euler condition

$$\frac{1}{c_t} = \beta (1+r) \frac{1}{c_{t+1}},$$

the goods consumption-leisure choice first-order condition,

$$w_t \frac{1}{c_t} = \ell_t^{-\gamma} \qquad \text{for} \qquad 1 > \ell_t > 0$$

and

$$w_t \frac{1}{c_t} \leq 1 \qquad \text{for} \qquad \ell_t = 1,$$

the leisure and labor constraints,

$$n_t + \ell_t \leq 1, \qquad n_t \geq 0 \qquad \text{and} \qquad \ell_t \geq 0,$$

the budget identity

$$a_{t+1} - a_t = ra_t + w_t n_t - c_t,$$

the transversality condition,

$$\lim_{T\to\infty} \beta^T \frac{\partial u\left(c_t, \ell_t\right)}{\partial c_t} a_{T+1} = 0,$$

and the solvency constraint, $\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T a_{T+1} \geq 0$, and initial condition, a_0.

c) Ans: First, we note that $\ell_t > 0$ and $n_t = 1 - \ell_t$ in the optimum because the marginal utility of leisure is always positive and goes to infinity as ℓ approaches zero for this utility function. The intertemporal budget constraint is found by solving the budget identity forward and imposing the transversality condition as

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t = (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t n_t.$$

The first-order condition for the labor-leisure choice,

$$w_t \frac{1}{c_t} = \ell^{-\gamma},$$

(assume an interior solution such that $n_t > 0$) gives us labor supply as

$$n_t = 1 - \left(\frac{c_t}{w_t}\right)^{\frac{1}{\gamma}}.$$

Substituting the Euler condition, $c_t = \beta^t (1+r)^t c_0$ and the labor supply equation into the budget constraint leads to

$$\begin{aligned}
(1-\beta)^{-1} c_0 &= (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t \left(1 - \left(\frac{\beta^t (1+r)^t c_0}{w_t}\right)^{\frac{1}{\gamma}}\right) \\
&= (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w_t - c_0^{\frac{1}{\gamma}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^{\left(1-\frac{1}{\gamma}\right)t} \beta^{\frac{1}{\gamma}t} w_t^{1-\frac{1}{\gamma}}.
\end{aligned}$$

For any time t, the consumption function is

$$(1-\beta)^{-1} c_0 = (1+r)\, a_0 + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} w_s - c_0^{\frac{1}{\gamma}} \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{\left(1-\frac{1}{\gamma}\right)(s-t)} \beta^{\frac{1}{\gamma}(s-t)} w_t^{1-\frac{1}{\gamma}}.$$

Using the constant wage rate, $w_t = w$, we have

$$\begin{aligned}
(1-\beta)^{-1} c_0 &= (1+r)\, a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t w - c_0^{\frac{1}{\gamma}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^{\left(1-\frac{1}{\gamma}\right)t} \beta^{\frac{1}{\gamma}t} w^{1-\frac{1}{\gamma}} \\
&= (1+r)\, a_0 + \frac{1+r}{r} w - w^{1-\frac{1}{\gamma}} c_0^{\frac{1}{\gamma}} \frac{1+r}{1+r-\left((1+r)\,\beta\right)^{\frac{1}{\gamma}}}.
\end{aligned}$$

This can be rearranged to

$$\frac{r}{1+r}\frac{1+\rho}{\rho} c_t + w^{1-\frac{1}{\gamma}} c_t^{\frac{1}{\gamma}} \frac{r}{1+r-\left((1+r)\,\beta\right)^{\frac{1}{\gamma}}} = r a_t + w,$$

for any $t \geq 0$. This is a little bit of a mess but gives us an implicit equation for c_t.

Beyond what is asked in the question, we can do a little better. Start with the original budget constraint written as

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}c_t+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}w\left(1-n_t\right)=\left(1+r\right)a_0+\frac{1+r}{r}w,$$

where the left-hand side is the present value of expenditures on consumption and leisure, $\ell = 1-n$, and the right-hand side is the value of the household's total endowment. The goods consumption-leisure choice equation and Euler condition combine to get an Euler condition in leisure consumption,

$$\left(1-n_t\right)^{\gamma}=\beta^{t}\left(1+r\right)^{t}\left(1-n_0\right)^{\gamma}.$$

Using the two results,

$$c_t=\beta^{t}\left(1+r\right)^{t}c_0 \qquad \text{and} \qquad 1-n_t=\left(\beta\left(1+r\right)\right)^{\frac{1}{\gamma}t}\left(1-n_0\right)$$

the budget constraint solves to get

$$\frac{1}{1-\beta}c_t+\frac{1+r}{1+r-\left(1+r\right)^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}}w\left(1-n_t\right)=\left(1+r\right)a_t+\frac{1+r}{r}w.$$

Lastly, we just need to use $\ell_t=\left(\frac{c_t}{w}\right)^{\frac{1}{\gamma}}$ to solve for the share of consumption of goods and of leisure in the total value of the endowment of the household, $W_t\equiv\left(1+r\right)a_t+\frac{1+r}{r}w$.

Just split the solution for c_t above into two parts as follows

$$\frac{1}{1-\beta}c_t=\psi W_t \qquad \text{and} \qquad \frac{1+r}{1+r-\left(1+r\right)^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}}w\left(1-n_t\right)=\left(1-\psi\right)W_t$$

where using $w\left(1-n_t\right)=w^{1-\frac{1}{\gamma}}c_t^{\frac{1}{\gamma}}$, we have that

$$\frac{1-\psi}{\psi^{\frac{1}{\gamma}}}=\frac{1+r}{1+r-\left(1+r\right)^{\frac{1}{\gamma}}\beta^{\frac{1}{\gamma}}}w_t^{1-\frac{1}{\gamma}}\left(1-\beta\right)^{\frac{1}{\gamma}}W_t^{\frac{1}{\gamma}-1}.$$

You should note the dependence of the share of the total endowment spent on goods on the value of the total endowment itelf through $W_t^{\frac{1}{\gamma}-1}$ unless $\gamma=1$.

d) Ans: This time, we substitute the specified wage rate sequence into the budget constraint to

$$\begin{aligned}\frac{1}{1-\beta}c_0 &= \left(1+r\right)a_0+\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}w+\sum_{t=T}^{\infty}\left(\frac{1}{1+r}\right)^{t}\Delta w\\ &\quad -c_0^{\frac{1}{\gamma}}\sum_{t=0}^{T-1}\left(\frac{1}{1+r}\right)^{\left(1-\frac{1}{\gamma}\right)t}\beta^{\frac{1}{\gamma}t}w^{1-\frac{1}{\gamma}}-c_0^{\frac{1}{\gamma}}\sum_{t=T}^{\infty}\left(\frac{1}{1+r}\right)^{\left(1-\frac{1}{\gamma}\right)t}\beta^{\frac{1}{\gamma}t}\left(w+\Delta w\right)^{1-\frac{1}{\gamma}}\\ &= \left(1+r\right)a_0+\left(\frac{1+r}{r}\right)\left(w+\left(\frac{1}{1+r}\right)^{T}\Delta w\right)\\ &\quad -c_0^{\frac{1}{\gamma}}\frac{1+r}{1+r-\beta^{\frac{1}{\gamma}}\left(1+r\right)^{\frac{1}{\gamma}}}\left(\left(1-\left(\frac{1}{1+r}\right)^{T}\right)w^{1-\frac{1}{\gamma}}+\left(\frac{1}{1+r}\right)^{T}\left(w+\Delta w\right)^{1-\frac{1}{\gamma}}\right).\end{aligned}$$

This equation solves for c_0 (but not in closed form), as

$$\begin{aligned} & \frac{1}{1-\beta}c_0 + c_0^{\frac{1}{\gamma}} \frac{1+r}{1+r-\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}} \left(\left(1 - \left(\frac{1}{1+r} \right)^T \right) w^{1-\frac{1}{\gamma}} + \left(\frac{1}{1+r} \right)^T (w+\Delta w)^{1-\frac{1}{\gamma}} \right) \\ = \; & (1+r)\, a_0 + \frac{1+r}{r} \left(w + \left(\frac{1}{1+r} \right)^T \Delta w \right). \end{aligned}$$

A little rearrangement leads to an equation in terms of consumption of goods and leisure,

$$\begin{aligned} & \frac{1}{1-\beta}c_0 + w(1-n_0) \frac{1+r}{1+r-\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}} \left(\left(1 - \left(\frac{1}{1+r} \right)^T \right) + \left(\frac{1}{1+r} \right)^T \left(\frac{w+\Delta w}{w} \right)^{1-\frac{1}{\gamma}} \right) \\ = \; & (1+r)\, a_0 + \frac{1+r}{r} \left(w + \left(\frac{1}{1+r} \right)^T \Delta w \right), \end{aligned}$$

using the first-order condition, $1 - n_0 = \frac{c_0}{w}^{\frac{1}{\gamma}}$.

Now, letting $r = \rho$, we have

$$\begin{aligned} & \frac{1+r}{r}c_0 + w(1-n_0) \frac{1+r}{r} \left(\left(1 - \left(\frac{1}{1+r} \right)^T \right) + \left(\frac{1}{1+r} \right)^T \left(\frac{w+\Delta w}{w} \right)^{1-\frac{1}{\gamma}} \right) \\ = \; & (1+r)\, a_0 + \frac{1+r}{r} \left(w + \left(\frac{1}{1+r} \right)^T \Delta w \right) \end{aligned}$$

which becomes

$$c_0 + w(1-n_0) \left(\left(1 - \left(\frac{1}{1+r} \right)^T \right) + \left(\frac{1}{1+r} \right)^T \left(\frac{w+\Delta w}{w} \right)^{1-\frac{1}{\gamma}} \right) = ra_0 + w + \left(\frac{1}{1+r} \right)^T \Delta w$$

The right-hand side of this last equation,

$$ra_0 + w + \left(\frac{1}{1+r} \right)^T \Delta w,$$

is household permanent income at $t = 0$, and the left-hand side,

$$c_0 + w(1-n_0) \left(\left(1 - \left(\frac{1}{1+r} \right)^T \right) + \left(\frac{1}{1+r} \right)^T \left(\frac{w+\Delta w}{w} \right)^{1-\frac{1}{\gamma}} \right),$$

is household expenditure on all consumption (that is, of both goods and leisure) at $t = 0$.

Household savings is found by substituting into

$$a_{t+1} - a_t = ra_t + w_t n_t - c_t$$

to get

$$a_1 - a_0 = -\left(\frac{1}{1+r} \right)^T \Delta w + w(1-n_0) \left(-\left(\frac{1}{1+r} \right)^T + \left(\frac{1}{1+r} \right)^T \left(\frac{w+\Delta w}{w} \right)^{1-\frac{1}{\gamma}} \right)$$

We can see that savings will fall at $t = 0$ with an anticipated future increase in w. This is because consumption expenditure (that is, $c_0 + w\ell_0$) rises with permanent income, while the current value of the endowment of time, w, does not rise. The budget identity and the implicit equation for consumption expenditures imply this.

4. a) Ans: The value of the firm is the present value of its cash flow (just call these dividends),

$$\begin{aligned} V_0 &= \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(y_t - w_t n_t - k_{t+1} + (1-\delta)\,k_t + b_{t+1} - (1+r)\,b_t\right) \\ &= \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(y_t - w_t n_t - k_{t+1} + (1-\delta)\,k_t\right) + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(b_{t+1} - (1+r)\,b_t\right) \end{aligned}$$

The second summation is

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(b_{t+1} - (1+r)\,b_t\right) = \lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t b_{t+1} - (1+r)\,b_0.$$

Imposing the solvency constraint with equality, the value of the firm is

$$V_0 = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \left(y_t - w_t n_t - k_{t+1} + (1-\delta)\,k_t\right) - (1+r)\,b_0.$$

b) Ans: The first-order conditions for employment of capital and labor by the firm are

$$f'\left(\frac{k_t}{n_t}\right) - \delta = r \qquad \text{and} \qquad f\left(\frac{k_t}{n_t}\right) - \frac{k_t}{n_t} f'\left(\frac{k_t}{n_t}\right) = w_t.$$

Substituting after noting that

$$y_t - w_t n_t = k_t f'\left(\frac{k_t}{n_t}\right) = (r+\delta)\,k_t,$$

we get

$$\begin{aligned} V_0 &= \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (1+r)\,k_t - k_{t+1} \\ &= (1+r)\,k_0 - \lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t k_{t+1} - (1+r)\,b_0 \end{aligned}$$

Imposing the constraint $k_{t+1} \geq 0$ and assuming (correctly, but not yet done) the transversality condition holds, this becomes

$$V_0 = (1+r)\,(k_0 - b_0)\,.$$

If the firm initially owns no capital, then it must borrow the initial capital, k_0, so that $k_0 - b_0 = 0$.

Now, we solve the optimization problem for the firm to get the transversality conditions. The problem can be written as

$$\max_{\{n_t, i_t, x_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (y_t - w_t n_t - i_t + x_t)$$

subject to the equation of motion for capital,

$$k_{t+1} = (1-\delta)\,k_t + i_t$$

and the budget identity for debt,

$$b_{t+1} = (1+r)\, b_t + x_t$$

given the initial conditions, k_0 and b_0, inequality constraint, $k_{t+1} \geq 0$ and solvency constraint, $\lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t b_{t+1}$. The variable x is net new borrowing by the firm.

The Lagrangian is

$$L = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (y_t - w_t n_t - i_t + x_t) + \lambda_t\left((1-\delta)\, k_t + i_t - k_{t+1}\right) + \mu_t\left((1+r)\, b_t + x_t - b_{t+1}\right)$$

where the control variables are n_t, i_t and x_t, and the state variables are k_t and b_t.

The necessary conditions for an optimum are with respect to n,

$$f\left(\frac{k_t}{n_t}\right) - \frac{k_t}{n_t} f'\left(\frac{k_t}{n_t}\right) = w_t,$$

with respect to i,

$$\left(\frac{1}{1+r}\right)^t = \lambda_t,$$

with respect to x,

$$\left(\frac{1}{1+r}\right)^t = \mu_t,$$

$$\lambda_t = (1-\delta)\, \lambda_{t+1} \qquad \text{and} \qquad \mu_t = (1+r)\, \mu_{t+1},$$

the transversality conditions,

$$\lim_{t\to\infty} \lambda_t b_{t+1} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \mu_t k_{t+1} = 0$$

and the solvency constraint, inequality of capital constraint and initial conditions.

We can see immediately that we correctly assumed that the transversality condtions are

$$\lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t b_{t+1} = 0 \qquad \text{and} \qquad \lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t k_{t+1} = 0.$$

The value of the firm is $V_0 = (1+r)\,(k_0 - b_0)$.

c) Ans: Begin with the value of the firm,

$$V_0 = \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (y_t - w_t n_t - \nu_t k_t)$$

and maximize this with respect to $\{n_t, k_t\}_{t=0}^{\infty}$. The first-order conditions are just the same as maximizing per period profit, $y_t - w_t n_t - \nu_t k_t$,

$$f'\left(\frac{k_t}{n_t}\right) = \nu_t \qquad \text{and} \qquad f\left(\frac{k_t}{n_t}\right) - \frac{k_t}{n_t} f'\left(\frac{k_t}{n_t}\right) = w_t.$$

The owners of capital maximize their value, V_0^k,

$$\max_{\{i_t, k_{t+1}\}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (\nu_t k_t - i_t)$$

subject to

$$k_{t+1} = (1-\delta) k_t + i_t$$

given the constraint, $k_{t+1} \geq 0$. The necessary conditions include

$$\left(\frac{1}{1+r}\right)^t = \lambda_t$$

and

$$\lambda_t = \left(\frac{1}{1+r}\right)^{t+1} \nu_{t+1} + (1-\delta) \lambda_{t+1}.$$

Putting these together, we get the intuitive result,

$$\left(\frac{1}{1+r}\right)^t = (\nu_{t+1} + (1-\delta)) \left(\frac{1}{1+r}\right)^{t+1} \qquad \Rightarrow \qquad \nu_{t+1} = r + \delta.$$

The opportunity cost of owning capital is $r + \delta$. This is the rental cost of capital (ν) in equilibrium.

The value of the firm is

$$\begin{aligned} V_0 &= \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (y_t - w_t n_t - (r+\delta) k_t) \\ &= \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (0) = 0. \end{aligned}$$

The only value of the firm is the initial capital, $(1+r) k_0$, its owners happen to own at time 0. They are indifferent between using this themselves or renting it to others. In parts a and b, they are indifferent between using it themselves or lending it to others.