Economics 205A
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Sample Answers for Problem Set 2

1. a) Ans: We have the linearized solution from Problem Set 1. The steady state, (k^*, c^*) , is given by

$$1 = \beta\left(1 + Af'\left(k^*\right) - \delta\right)$$

and

$$c^* = Af\left(k^*\right) - \delta k^* .$$

The linearized dynamics are given by

$$\begin{bmatrix} c_{t+1} - c_t \\ k_{t+1} - k_t \end{bmatrix} = \begin{bmatrix} \beta\frac{u'(c^*)}{u''(c^*)}Af''\left(k^*\right) & -\frac{u'(c^*)}{u''(c^*)}Af''\left(k^*\right) \\ -1 & \rho \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix},$$

and the convergent saddle-path solution is given by

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = \left(k_1 - k^*\right) \begin{bmatrix} \rho - \lambda_- \\ 1 \end{bmatrix} \left(1 + \lambda_-\right)^t ,$$

where

$$\lambda_- = \frac{1}{2}\left[\left(\beta\frac{u'\left(c^*\right)}{u''\left(c^*\right)}Af''\left(k^*\right) + \rho\right) - \sqrt{\left(-\beta\frac{u'\left(c^*\right)}{u''\left(c^*\right)}Af''\left(k^*\right) - \rho\right)^2 + 4\beta\frac{u'\left(c^*\right)}{u''\left(c^*\right)}Af''\left(k^*\right)}\right] .$$

The convergent saddle-path solution is written for an initial capital stock, k_1 , at time 1 . k_1 needs to be determined. k_0 is given at time 0 .

In part d, we will use the slope of the saddle-path solution for $t \geq 1$, which is

$$\frac{c_t - c^*}{k_t - k^*} = \rho - \lambda_- .$$

b) Ans: The resource identity at time 0 is

$$\Delta k_1 \equiv k_1 - k_0 = (A + \Delta A) f\left(k_0\right) - \delta k_0 - c_0 .$$

The economy begins in the steady state for $y = Af(k)$, (k^*, c^*) , so that $k_0 = k^*$ and $0 = Af\left(k^*\right) - \delta k^* - c^*$, so that

$$\Delta k_1 = \Delta A f\left(k^*\right) - \left(c_0 - c^*\right)$$

c) Ans: The Euler condition at $t = 0$ is

$$u'(c_0) = (1 + Af'(k_1) - \delta)\,\beta u'(c_1)\ .$$

Linearized this is

$$u''(c^*)(c_0 - c^*) = u''(c^*)(c_1 - c^*) + \beta u'(c^*)\,Af''(k^*)(k_1 - k^*)$$

d) Ans: The three equations we use are the linearized resource identity,

$$k_1 - k^* = \Delta A f(k^*) - (c_0 - c^*)$$

the linearized Euler condition,

$$c_0 - c^* = (c_1 - c^*) + \beta \frac{u'(c^*)}{u''(c^*)} Af''(k^*)(k_1 - k^*)$$

and the stable eigenvector slope,

$$\frac{c_1 - c^*}{k_1 - k^*} = \rho - \lambda_-\ .$$

The last equation is used because the optimum coincides with the stable saddle path beginning at date $t = 1$. Solving, we get

$$k_1 - k^* = \left[\frac{1}{1 + \rho - \lambda_- + \beta\frac{u'(c^*)}{u''(c^*)}Af''(k^*)}\right] f(k^*)\,\Delta A$$

$$c_0 - c^* = \left[\frac{\rho - \lambda_- + \beta\frac{u'(c^*)}{u''(c^*)}Af''(k^*)}{1 + \rho - \lambda_- + \beta\frac{u'(c^*)}{u''(c^*)}Af''(k^*)}\right] f(k^*)\,\Delta A$$

and

$$c_1 - c^* = \left[\frac{\rho - \lambda_-}{1 + \rho - \lambda_- + \beta\frac{u'(c^*)}{u''(c^*)}Af''(k^*)}\right] f(k^*)\,\Delta A$$

We can see that $\frac{\Delta k_1}{\Delta A} > 0$, $\frac{c_0 - c^*}{\Delta A} > 0$ and $\frac{c_1 - c^*}{\Delta A} > 0$. Checking, we correctly have that $\frac{\Delta k_1}{\Delta A} + \frac{c_0 - c^*}{\Delta A} = f(k^*)$ and that c_0 and c_1 satisfy the linearized Euler condition,

$$c_0 - c_1 = \beta \frac{u'(c^*)}{u''(c^*)} Af''(k^*)\,\Delta k_1 > 0.$$

e) Ans: The proper phase diagram shows an upward sloping linear approximation of the saddle path through the steady state. Mark c_0 straight above $k_0 = k^*$ and a point $k_1 > k^*$. c_1 is given by the point (k_1, c_1) that lies on the saddle path, and $k_1 + c_0 = (A + \Delta A) f(k^*) - \delta k^*$. The reason the claim on p. 32 is incorrect is that if the increase in current resources, $\Delta A f(k^*)$, were completely consumed at $t = 0$, the Euler condition would be violated: $u'(c^* + \Delta A f(k^*)) \neq u'(c^*)\,\beta(1 + Af'(k^*) - \delta)$ because $\beta(1 + Af'(k^*) - \delta) = 1$ in the steady state.

2. a) Ans: The planner solves

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$k_{t+1} = (1 + A - \delta) k_t - c_t \ ,$$

the constraints, $k_t \geq 0$ for all $t \geq 0$ and the initial condition that k_0 is given.

To solve, set up the Lagrangian,

$$L = \left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left((1 + A - \delta) k_t - c_t - k_{t+1} \right) \right]$$

and find the first-order conditions,

$$\beta^t c_t^{-\sigma} = \lambda_t \ ,$$

$$\lambda_{t+1} = (1 + A - \delta) \lambda_t$$

and

$$k_{t+1} = (1 + A - \delta) k_t - c_t \ ,$$

plus the transversality condition,

$$\lim_{t \to \infty} \lambda_t k_{t+1} = 0.$$

We also have the constraints, $k_t \geq 0$ and $c_t \geq 0$ for all $t \geq 0$.

The solutions are found through the steps:

$$c_t^{-\sigma} = \beta (1 + A - \delta) c_{t+1}^{-\sigma} \qquad \Rightarrow \qquad c_{t+1} = \beta^{\frac{1}{\sigma}} (1 + A - \delta)^{\frac{1}{\sigma}} c_t$$

which together with

$$k_{t+1} = (1 + A - \delta) k_t - c_t$$

gives the dynamics

$$\begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} \beta^{\frac{1}{\sigma}} (1 + A - \delta)^{\frac{1}{\sigma}} & 0 \\ -1 & (1 + A - \delta) \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} .$$

The eigenvalues are

$$\lambda_1 = \beta^{\frac{1}{\sigma}} (1 + A - \delta)^{\frac{1}{\sigma}} \qquad \text{and} \qquad \lambda_2 = 1 + A - \delta$$

and the associated eigenvectors are

$$\nu_1 = \begin{bmatrix} \lambda_2 - \lambda_1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \nu_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

The solution that satisfies the transversality condition is

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \left(\beta^{\frac{1}{\sigma}} (1 + A - \delta)^{\frac{1}{\sigma}} \right)^t \begin{bmatrix} c_0 \\ k_0 \end{bmatrix}$$

which you can verify by

$$
\begin{aligned}
\lim_{t\to\infty} \beta^t c_t^{-\sigma} k_{t+1} &= \lim_{t\to\infty} \beta^t \left(\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}}\right)^{-\sigma t} c_0^{-\sigma} \left(\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}}\right)^{t+1} k_0 \\
&= c_0^{-\sigma} k_0 \lim_{t\to\infty} \beta^t \left(\beta (1+A-\delta)\right)^{-t} \left(\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}}\right)^{t+1} \\
&= c_0^{-\sigma} (1+A-\delta) k_0 \lim_{t\to\infty} \left(\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}-1}\right)^{t+1} = 0
\end{aligned}
$$

as long as $\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}-1} < 1$.

We could solve for this differently (and equivalently) solving forward the Euler condition and the resource identity as

$$
c_t = \left[\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}}\right]^t c_0 \ ,
$$

$$
\lim_{T\to\infty} (1+A-\delta)^{-T} k_{T+1} - k_0 = -\sum_{t=0}^{\infty} (1+A-\delta)^{-t} c_t \ .
$$

The transversality condition,

$$
\lim_{t\to\infty} \beta^t c_t^{-\sigma} k_{t+1} = 0
$$

implies

$$
\lim_{t\to\infty} \beta^t c_0^{-\sigma} \left(\beta (1+A-\delta)\right)^{-t} k_{t+1} = 0 \qquad \Rightarrow \qquad c_0^{-\sigma} \lim_{t\to\infty} (1+A-\delta)^{-t} k_{t+1} = 0
$$

so that $\lim_{T\to\infty} (1+A-\delta)^{-T} k_{T+1} = 0$ and

$$
\begin{aligned}
k_0 &= \sum_{t=0}^{\infty} (1+A-\delta)^{-t} c_t \\
&= \sum_{t=0}^{\infty} (1+A-\delta)^{-t} \left[\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}}\right]^t c_0 \\
&= \frac{1}{1-\left[\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}-1}\right]} c_0
\end{aligned}
$$

when the series $\sum_{t=0}^{\infty} (1+A-\delta)^{-t} \left[\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}}\right]^t$ converges which requires that $\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}-1} < 1$
.

b) Ans: The condition is derived just above:

$$
\beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}-1} < 1
$$

which can be rearranged to

$$
1+A-\delta - \beta^{\frac{1}{\sigma}} (1+A-\delta)^{\frac{1}{\sigma}} > 0
$$

which implies $\lambda_1 < \lambda_2$. This is the condition for the transversality condition to be satisfied for $c_0 > 0$ given $k_0 > 0$. It is also the necessary condition for the intertemporal resource constraint to be satisfied as shown at the end of the answer to part a.

c) Ans: Household utility in the optimum is given by

$$\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}-1}{1-\sigma} = \sum_{t=0}^{\infty}\beta^t\frac{\left[\beta^{\frac{1}{\sigma}}(1+A-\delta)^{\frac{1}{\sigma}}\right]^{(1-\sigma)t}c_0^{1-\sigma}-1}{1-\sigma}$$

$$= \frac{c_0^{1-\sigma}}{1-\sigma}\sum_{t=0}^{\infty}\beta^t\left[\beta^{\frac{1}{\sigma}}(1+A-\delta)^{\frac{1}{\sigma}}\right]^{(1-\sigma)t}-\frac{1}{1-\sigma}\sum_{t=0}^{\infty}\beta^t$$

$$= \frac{c_0^{1-\sigma}}{1-\sigma}\left(1-\beta^{\frac{1}{\sigma}}(1+A-\delta)^{\frac{1-\sigma}{\sigma}}\right)^{-1}-\frac{1}{1-\sigma}\frac{1}{1-\beta},$$

where the series converges if

$$1-\beta^{\frac{1}{\sigma}}(1+A-\delta)^{\frac{1-\sigma}{\sigma}}>0$$

which can be rewritten as the same condition as in the answer to part b,

$$A-\delta+1-\beta^{\frac{1}{\sigma}}(1+A-\delta)^{\frac{1}{\sigma}}>0.$$

3. Ans: a) The problem for finding the command optimum is

$$\max_{\{c_t,\ell_t,n_t,k_{t+1}\}_{t-0}^{\infty}}\sum_{t=0}^{\infty}\beta^t u(c_t,\ell_t)$$

subject to the budget identity

$$k_{t+1}-k_t=Af(k_t,n_t)-\delta k_t-c_t\ ,$$

the leisure endowment constraint,

$$n_t+\ell_t\leq 1\ ,$$

and the non-negativity constraints for labor supply, leisure consumption, and capital,

$$n_t\geq 0,\quad \ell_t\geq 0\quad\text{and}\quad k_t\geq 0,\quad\text{for all }\ t\geq 0\ ,$$

given initial capital k_0 .

The necessary conditions for an optimum are derived from the Lagrangian

$$L=\sum_{t=0}^{\infty}\left[\beta^t u(c_t,\ell_t)+\lambda_t\left(Af(k_t,n_t)+(1-\delta)k_t-c_t-k_{t+1}\right)+\eta_t^1\ell_t+\eta_t^2 n_t+\gamma_t(1-n_t-\ell_t)+\mu_t k_{t+1}\right]\ .$$

We will assume that $\frac{\partial u(c_t,\ell_t)}{\partial c_t}$ and $\frac{\partial u(c_t,\ell_t)}{\partial \ell_t}$ are positive for all $c_t>0$ and $\ell_t>0$.

The necessary conditions are:

the Euler condition

$$\frac{\partial u(c_t,\ell_t)}{\partial c_t}=\beta\left(1+A\frac{\partial f(k_{t+1},n_{t+1})}{\partial k_{t+1}}-\delta\right)\frac{\partial u(c_{t+1},\ell_{t+1})}{\partial c_{t+1}}\ ,$$

the labor-leisure choice first-order condition,

$$A\frac{\partial f\left(k_t,n_t\right)}{\partial n_t}\frac{\partial u\left(c_t,\ell_t\right)}{\partial c_t}=\frac{\partial u\left(c_t,\ell_t\right)}{\partial \ell_t}\qquad\text{for}\qquad 1>\ell_t>0\ ,$$

with

$$A\frac{\partial f\left(k_t,n_t\right)}{\partial n_t}\frac{\partial u\left(c_t,\ell_t\right)}{\partial c_t}\geq\frac{\partial u\left(c_t,\ell_t\right)}{\partial \ell_t}\qquad\text{for } \ell_t=0\qquad\text{and}\qquad A\frac{\partial f\left(k_t,n_t\right)}{\partial n_t}\frac{\partial u\left(c_t,\ell_t\right)}{\partial c_t}\leq\frac{\partial u\left(c_t,\ell_t\right)}{\partial \ell_t}\qquad\text{for } \ell_t=1$$

(an Inada condition for $\frac{\partial f(k_t,n_t)}{\partial n_t}$ will eliminate the second corner condition), the leisure and labor constraints,

$$n_t+\ell_t\leq 1,\qquad n_t\geq 0\qquad\text{and}\qquad \ell_t\geq 0\ ,$$

the resource identity

$$k_{t+1}-k_t=Af\left(k_t,n_t\right)-\delta k_t-c_t\ ,$$

the transversality condition,

$$\lim_{t\to\infty}\beta^t\frac{\partial u\left(c_t,\ell_t\right)}{\partial c_t}k_{t+1}=0\ ,$$

and the inequality constraint, $\lim_{t\to\infty}k_{t+1}\geq 0$ and initial condition, k_0 .

b) Ans: The necessary conditions become

$$\frac{c_{t+1}}{c_t}=\beta\left(1+A\frac{\partial f\left(k_{t+1},n_{t+1}\right)}{\partial k_{t+1}}-\delta\right)\ ,$$

the labor-leisure choice first-order condition,

$$\frac{c_t}{\ell_t^{\gamma}}=A\frac{\partial f\left(k_t,n_t\right)}{\partial n_t}\qquad\text{for}\qquad 1>\ell_t>0\ ,$$

so that $\ell_t>0$ always, and assume $\lim_{n\to 0}\frac{\partial f(k,n)}{\partial n}=\infty$ to eliminate $n_t=0$, the resource identity is the same,

$$k_{t+1}-k_t=Af\left(k_t,n_t\right)-\delta k_t-c_t\ ,$$

and the transversality condition,

$$\lim_{t\to\infty}\beta^t\frac{1}{c_t}k_{t+1}=0\ ,$$

and $\lim_{t\to\infty}k_{t+1}\geq 0$ hold.

c) Ans: The steady state, (k^*,c^*,n^*) is given by

$$A\frac{\partial f\left(k^*,n^*\right)}{\partial k}-\delta=\rho\ ,$$

$$Af\left(k^*,n^*\right)-\delta k^*=c^*$$

and

$$c^*=\left(1-n^*\right)^{\gamma}A\frac{\partial f\left(k^*,n^*\right)}{\partial n}\ .$$

Differentiating we get

$$A\frac{\partial^2 f(k^*,n^*)}{\partial k^2}dk + A\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n}dn + \frac{\partial f(k^*,n^*)}{\partial k}dA = 0\ ,$$

$$A\frac{\partial f(k^*,n^*)}{\partial k}dk - \delta dk + A\frac{\partial f(k^*,n^*)}{\partial n}dn + f(k^*,n^*)\,dA = dc$$

and

$$dc = (1-n^*)^\gamma\left(\frac{\partial f(k^*,n^*)}{\partial n}dA + A\frac{\partial^2 f(k^*,n^*)}{\partial n^2}dn + A\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n}dk\right) - \gamma(1-n^*)^{\gamma-1}A\frac{\partial f(k^*,n^*)}{\partial n}dn\ .$$

Substituting first-order conditions to simplify, we can use

$$A\frac{\partial^2 f(k^*,n^*)}{\partial k^2}dk + A\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n}dn = -\frac{\rho+\delta}{A}dA\ ,$$

$$\rho dk + \frac{c^*}{(1-n^*)^\gamma}dn - dc = -f(k^*,n^*)\,dA$$

and

$$(1-n^*)^\gamma\left(A\frac{\partial^2 f(k^*,n^*)}{\partial n^2}dn + A\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n}dk\right) - \gamma\frac{c^*}{1-n^*}dn - dc = -\frac{c^*}{A}dA$$

to solve for the three derivatives from

$$\begin{bmatrix} \frac{dk}{dA} \\ \frac{dn}{dA} \\ \frac{dc}{dA} \end{bmatrix} = \begin{bmatrix} A\frac{\partial^2 f(k^*,n^*)}{\partial k^2} & A\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n} & 0 \\ \rho & \frac{c^*}{(1-n^*)^\gamma} & -1 \\ (1-n^*)^\gamma A\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n} & (1-n^*)^\gamma A\frac{\partial^2 f(k^*,n^*)}{\partial n^2} - \gamma\frac{c^*}{1-n^*} & -1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\rho+\delta}{A} \\ -f(k^*,n^*) \\ -\frac{c^*}{A} \end{bmatrix}\ .$$

If we use these along with strict concavity of $f(k,n)$ (which requires that $\frac{\partial^2 f(k^*,n^*)}{\partial k^2} < 0$, $\frac{\partial^2 f(k^*,n^*)}{\partial n^2} > 0$ and $\frac{\partial^2 f(k^*,n^*)}{\partial k^2}\frac{\partial^2 f(k^*,n^*)}{\partial n^2} - \left(\frac{\partial^2 f(k^*,n^*)}{\partial k\partial n}\right)^2 > 0$) we get what we can surmise from the three undifferentiated equations and concavity: k^* rises and/or n^* falls to keep $A\frac{\partial f(k^*,n^*)}{\partial k} = \rho+\delta$, c^* rises with output net of depreciation, $c^* = Af(k^*,n^*) - \delta k^*$, and n^* falls so that the marginal rate of substitution of leisure for consumption must equal the marginal product of labor. (It is not hard to sign the elements of the inverted matrix multiplying the negative column vector.)

d) Ans: If the economy begins with capital below steady-state capital, the economy will converge on a saddle-path stable dynamic path toward the steady state. k is the single pre-determined (that is, state) variable while both c_t and n_t are forward looking variables. However, there is only one initial condition, k_0 , and one transversality condition,

$$\lim_{t\to\infty}\beta^t\frac{1}{c_t}k_{t+1} = 0.$$

We note that labor supply equilibrium,

$$\frac{c_t}{(1-n_t)^\gamma} = A\frac{\partial f(k_t,n_t)}{\partial n_t}\ ,$$

determines n_t as a function of k_t and c_t only. Thus, we substitute the implicit function $n_t = \phi(k_t,c_t)$ into the Euler condition and the resource identity to get a two-dimensional dynamical system in k and c . A unique solution for $\{k_t, c_t\}_{t=0}^{\infty}$ will be determined given k_0 and the tranversality condition for $y_t = f(k_t, n_t)$ strictly concave. I did not expect you to do a lot more algebra, but if we do, we will get that c_t is rising with k_t . In general, the relationship between leisure consumption (equivalently, labor supply) will depend on the elasticity of substitution between capital and labor in production compared to the elasticity of substitution between consumption and leisure. Without a specific production function, you can say that all three converge monotonically to the steady state from an initial point near the steady state.

4. a) Ans: The two linearized equations derived in the text are

$$(1-\beta)(q_t - q^*) - \beta f''(k^*)(k_t - k^*) = \beta(q_{t+1} - q_t) + \beta f''(k^*)(k_{t+1} - k_t)$$

$$k^*(q_t - q^*) = \phi(k_{t+1} - k_t) \ ,$$

where we use the notation, $dq_t \equiv q_t - q^*$, as before.

All we do is substitute the second into the first to get

$$q_{t+1} - q_t = \left(\rho - \frac{k^*}{\phi} f''(k^*)\right)(q_t - q^*) - f''(k^*)(k_t - k^*)$$

and

$$k_{t+1} - k_t = \frac{k^*}{\phi}(q_t - q^*)$$

for the linearized system.

Writing first in matrix form,

$$\begin{bmatrix} q_{t+1} - q_t \\ k_{t+1} - k_t \end{bmatrix} = \begin{bmatrix} \left(\rho - \frac{f''(k^*)k^*}{\phi}\right) & -f''(k^*) \\ \frac{k^*}{\phi} & 0 \end{bmatrix} \begin{bmatrix} q_t - q^* \\ k_t - k^* \end{bmatrix} ,$$

we see that the matrix has a negative determinant, $\frac{k^*}{\phi} f''(k^*)$, so that there are two eigenvalues, one positive and one negative. The eigenvalues solve

$$\lambda_{\pm} = \frac{1}{2}\left(\rho - \frac{k^*}{\phi} f''(k^*) \pm \left(\left(\rho - \frac{k^*}{\phi} f''(k^*)\right)^2 - 4\frac{k^*}{\phi} f''(k^*)\right)^{1/2}\right) .$$

One solution is $\lambda_- < 0$ and the other is $\lambda_+ > \rho - \frac{f''(k^*)k^*}{\phi} > 0$. The eigenvectors satisfy

$$\nu_{\pm} = \begin{bmatrix} \frac{\phi}{k^*}\lambda_{\pm} \\ 1 \end{bmatrix} .$$

The slope of the saddle path that converges to the steady state is negative, and the slope of the divergent saddle path is positive.

Note that the locus, $\Delta k_{t+1} = 0$, has zero slope and the locus, $\Delta q_{t+1} = 0$, has negative slope. The slope of the (linearized) stable saddle path is between these. This can be seen by comparing the slope of

$\Delta q_{t+1} = 0$, $\frac{dq_t}{dk_t} = \frac{f''(k^*)}{\rho - \frac{f''(k^*)k^*}{\phi}}$, to the slope of the saddle path, $\frac{\phi}{k^*}\lambda_-$,

$$\frac{\frac{k^*}{\phi}f''(k^*)}{\rho - \frac{k^*}{\phi}f''(k^*)} < \frac{1}{2}\left(\rho - \frac{k^*}{\phi}f''(k^*) - \left(\left(\rho - \frac{k^*}{\phi}f''(k^*)\right)^2 - 4\frac{k^*}{\phi}f''(k^*)\right)^{1/2}\right)$$

To prove the inequality, rearrange and square both sides (remember to reverse the inequality symbol since each side is a negative number) gives us

$$\begin{aligned}
&\left(2\frac{\frac{k^*}{\phi}f''(k^*)}{\rho - \frac{k^*}{\phi}f''(k^*)} - \left(\rho - \frac{k^*}{\phi}f''(k^*)\right)\right)^2 \\
= \quad & \left(\rho - \frac{k^*}{\phi}f''(k^*)\right)^2 - 4\left(\rho - \frac{k^*}{\phi}f''(k^*)\right)\frac{\frac{k^*}{\phi}f''(k^*)}{\rho - \frac{k^*}{\phi}f''(k^*)} + 4\left(\frac{\frac{k^*}{\phi}f''(k^*)}{\rho - \frac{k^*}{\phi}f''(k^*)}\right)^2 \\
= \quad & \left(\rho - \frac{k^*}{\phi}f''(k^*)\right)^2 - 4\frac{k^*}{\phi}f''(k^*) + 4\left(\frac{\frac{k^*}{\phi}f''(k^*)}{\rho - \frac{k^*}{\phi}f''(k^*)}\right)^2 \\
> \quad & \left(\rho - \frac{k^*}{\phi}f''(k^*)\right)^2 - 4\frac{k^*}{\phi}f''(k^*) \ .
\end{aligned}$$

b) Ans: Rewrite the production function as

$$y_t = Af(k_t)$$

and just assume $A = 1$. The steady-state value of q is $q^* = 1 + \phi\delta$, and steady-state capital is given by

$$Af'(k^*) = \rho + \delta + \phi\delta\left(\rho + \frac{1}{2}\delta\right) \ .$$

Steady-state q^* does not change, and the change in k^* is

$$\frac{dk^*}{dA} = \frac{-f'(k^*)}{Af''(k^*)} > 0.$$

These shifts in the steady state and the slope of the stable eigenvector tell us that for k_0 , q_0 rises with A . As q_0 increases, so does i_0 reducing consumption c_0 .

c) Ans: This is similar to problem 2, but adjustment costs make a significant difference for the response of investment and consumption to a temporary productivity disturbance. Output at $t = 0$ rises by $\Delta y_0 = f(k^*)\Delta A$. Using the resource identity,

$$c_t = Af(k_t) - i_t\left(1 + \frac{\phi}{2}\frac{i_t}{k_t}\right)$$

and substituting the first-order condition for i_t , $q_t = 1 + \phi\frac{i_t}{k_t}$, leads to

$$c_t = Af(k_t) - k_t\left(\frac{q_t - 1}{\phi} + \frac{1}{2\phi}(q_t - 1)^2\right) \ .$$

Linearizing this version of the resource identity about the steady state gives

$$c_t - c^* = f(k_t)\,\Delta A + \left(Af'(k^*) - \frac{q^*-1}{\phi} - \frac{1}{2\phi}(q^*-1)^2\right)(k_t - k^*) - \frac{k^*}{\phi}q^*(q_t - q^*)\ .$$

After substituting the steady-state values, $q^* = 1 + \phi\delta$ and $Af'(k^*) = \rho(1+\phi\delta) + \delta\left(1 + \frac{\phi\delta}{2}\right)$, we get

$$c_t - c^* = f(k^*)\,\Delta A + \rho(1+\phi\delta)(k_t - k^*) - \left(\delta + \frac{1}{\phi}\right)k^*(q_t - q^*)\ .$$

This equation shows how the change in consumption and Tobin's q are related to ΔA at time t when productivity improves since $k_t = k^*$ is predetermined. That is,

$$c_t - c^* = f(k^*)\,\Delta A - \left(\delta + \frac{1}{\phi}\right)k^*(q_t - q^*)\ .$$

Next, we need to modify the equations of motion for k and q at time t to allow for the disturbance to A at time t . From the original problem, we have the first-order conditions,

$$\frac{u'(c_t)}{\beta u'(c_{t+1})}q_t = (1-\delta)\,q_{t+1} + \left(Af'(k_{t+1}) + \frac{\phi}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2\right)$$

and

$$k_{t+1} = (1-\delta)\,k_t + i_t\ .$$

Allowing A to change at time t , and using the approximation in Wickens, the linearized versions of these become

$$\beta(q_{t+1} - q_t) + \beta Af''(k^*)(k_{t+1} - k^*) = (1-\beta)(q_t - q^*) - \beta Af''(k^*)(k_t - k^*) - \beta f'(k^*)\,dA$$

and

$$(k_{t+1} - k_t) = \frac{k^*}{\phi}(q_t - q^*)\ .$$

Only the first of these equations changes. For future times $s > t$, the equations of motion are back to the originals,

$$\beta(q_{s+1} - q_s) + \beta Af''(k^*)(k_{s+1} - k_s) = (1-\beta)(q_s - q^*) - \beta Af''(k^*)(k_s - k^*)$$

$$(k_{s+1} - k_s) = \frac{k^*}{\phi}(q_s - q^*)\ ,$$

and the optimal solutions for q_s and k_s converge back to the original steady state along the saddle path solved in part b, $q_s - q^* = \frac{\phi}{k^*}\lambda_-(k_s - k^*)$. Substituting in $k_t = k^*$, we have that

$$\beta(q_{t+1} - q_t) + \beta Af''(k^*)(k_{t+1} - k^*) = (1-\beta)(q_t - q^*) - \beta f'(k^*)\,dA$$

and

$$(k_{t+1} - k^*) = \frac{k^*}{\phi}(q_t - q^*) \ .$$

Along with the linearized resource identity at time t ,

$$c_t - c^* = f(k^*)\Delta A - \left(\delta + \frac{1}{\phi}\right) k^* (q_t - q^*) \ ,$$

and the saddle-path solution,

$$q_{t+1} - q^* = \frac{\phi}{k^*}\lambda_- (k_{t+1} - k^*)$$

these give us the four conditions needed to solve for q_t , q_{t+1} , k_{t+1} and c_t given the temporary productivity disturbance ΔA .

The solutions are

$$q_t - q^* = \frac{f'(k^*)}{\rho - \lambda_- - A\frac{k^*}{\phi}f''(k^*)} dA \ ,$$

$$(k_{t+1} - k^*) = \frac{k^*}{\phi}(q_t - q^*) \ ,$$

and

$$\begin{aligned} c_t - c^* &= f(k^*)\Delta A - (\delta + \frac{1}{\phi})k^*(q_t - q^*) \\ &= \left[f(k^*) - \frac{(\delta + \frac{1}{\phi})k^* f'(k^*)}{\rho - \lambda_- - A\frac{k^*}{\phi}f''(k^*)}\right]\Delta A \ . \end{aligned}$$

Checking signs, we immediately see that $q_t > q^*$ and $k_{t+1} > k^*$, but $c_t - c^*$ cannot be signed from just looking at the expression. However, a couple substitutions using $k_{t+1} - k^* = \frac{k^*}{\phi}(q_t - q^*)$ and $k_{t+1} - k_t = k_{t+1} - k^* = i_t - \delta k^*$ lead us to $\delta k^* (q_t - q^*) = \delta\phi(i_t - i^*)$. Using this expression, we have

$$\begin{aligned} c_t - c^* &= f(k^*)\Delta A - \left(\delta + \frac{1}{\phi}\right) k^*(q_t - q^*) \\ &= f(k^*)\Delta A - (1 + \delta\phi)(i_t - i^*) \ . \end{aligned}$$

If we evaluate the cost of the rise in investment at time t as a perpetual expense, we get that the cost of $i_t - i^* = \Delta i_t$ is $\delta\phi\Delta i_t$ (that is, the marginal cost of maintaining the capital stock at k_{t+1} over k^*). If the increase in productivity lasted forever, the net benefit of raising the capital stock permanently to k_{t+1} would just be $f(k^*)\Delta A - (1 + \delta\phi)(i_t - i^*)$. This should be positive. If it were not, then the optimal $i_t - i^*$ would be smaller so that it did hold.