Economics 205A
Fall 2016
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Problem Set 1: Sample answers

1. (a) Ans: Write the Lagrangian as

$$L = \sum_{t=0}^{T} \left[\beta^t u(c_t) + \lambda_t \left((1-\delta) k_t + f(k_t) - c_t - k_{t+1}\right)\right]$$

and impose the restrictions that $k_{t+1} \geq 0$ and $c_t \geq 0$ for $T \geq t \geq 0$. The necessary conditions are

$$\frac{\partial L}{\partial c_t} = \beta^t u'(c_t) - \lambda_t \leq 0, \quad c_t \geq 0 \quad \text{and} \quad c_t \frac{\partial L}{\partial c_t} = 0,$$

for $T \geq t \geq 0$,

$$\frac{\partial L}{\partial \lambda_t} = (1-\delta) k_t + f(k_t) - c_t - k_{t+1} = 0$$

for $T \geq t \geq 0$,

$$\frac{\partial L}{\partial k_{t+1}} = \lambda_{t+1} \left(1 + f'(k_{t+1}) - \delta\right) - \lambda_t \leq 0, \quad k_{t+1} \geq 0 \quad \text{and} \quad k_{t+1} \frac{\partial L}{\partial k_{t+1}} = 0,$$

for $T - 1 \geq t \geq 0$, and

$$\frac{\partial L}{\partial k_{T+1}} = -\lambda_T \leq 0, \quad k_{T+1} \geq 0 \quad \text{and} \quad k_{T+1} \frac{\partial L}{\partial k_{T+1}} = 0.$$

Making the assumption that $\lim_{c \to 0} u'(c) = \infty$ and that $k_0 > 0$, these become

$$\frac{\partial L}{\partial c_t} = 0 \quad \Rightarrow \quad \beta^t u'(c_t) = \lambda_t > 0,$$

for $T \geq t \geq 0$,

$$\frac{\partial L}{\partial k_{t+1}} = 0 \quad \Rightarrow \quad \lambda_{t+1} \left(1 + f'(k_{t+1}) - \delta\right) - \lambda_t = 0,$$

for $T - 1 \geq t \geq 0$,

$$k_{t+1} = (1-\delta) k_t + f(k_t) - c_t,$$

for $T \geq t \geq 0$, and

$$\lambda_T k_{T+1} = 0 \quad \text{and} \quad k_{T+1} \geq 0.$$

These can be simplified to

$$u'(c_t) = \left(1 + f'(k_{t+1}) - \delta\right) \beta u'(c_{t+1})$$

for $T - 1 \geq t \geq 0$,

$$k_{t+1} = (1-\delta) k_t + f(k_t) - c_t,$$

for $T \geq t \geq 0$, and

$$\beta^T u'(c_T) k_{T+1} = 0 \quad \text{and} \quad k_{T+1} \geq 0.$$

(b) Ans: The necessary conditions become

$$u'(c_t) = \left(1 + f'(k_{t+1}) - \delta\right) \beta u'(c_{t+1})$$

for all $t \geq 0$,

$$k_{t+1} = (1 - \delta) k_t + f(k_t) - c_t,$$

for all $t \geq 0$, and

$$\lim_{t \to \infty} \beta^T u'(c_T) k_{T+1} = 0 \quad \text{and} \quad \lim_{T \to \infty} k_{T+1} \geq 0.$$

The only change is in the range for t, hence the limits with respect to T.

(c) Ans: The steady state satisfies

$$c_{t+1} = c_t \quad \text{and} \quad k_{t+1} = k_t.$$

Thus, the steady-state values of c and k are determined by

$$1 = \beta \left(1 + f'(k^*) - \delta\right)$$

and

$$c^* = f(k^*) - \delta k^*.$$

(d) Ans: The dynamics are given by

$$\frac{c_{t+1}}{c_t} = \beta \left(1 + f'(k_{t+1}) - \delta\right)$$

and

$$k_{t+1} = (1 - \delta) k_t + f(k_t) - c_t,$$

for all $t \geq 0$.

Linearizing these leads to

$$c_{t+1} - c_t = \beta c^* f''(k^*)(k_t - k^*) + \beta c^* f''(k^*)(k_{t+1} - k_t)$$

and

$$k_{t+1} - k_t = \left(f'(k^*) - \delta\right)(k_t - k^*) - (c_t - c^*).$$

Substituting the linearized resource identity (the second equation) into the linearized Euler condition (the first equation), we get the system of two difference equations,

$$c_{t+1} - c_t = \beta c^* f''(k^*) \beta^{-1} (k_t - k^*) - \beta c^* f''(k^*)(c_t - c^*)$$

and

$$k_{t+1} - k_t = \rho (k_t - k^*) - (c_t - c^*),$$

where $\rho = \beta^{-1} - 1$. Writing this in matrix form, the linearized system is

$$\begin{bmatrix} c_{t+1} - c_t \\ k_{t+1} - k_t \end{bmatrix} = \begin{bmatrix} -\beta c^* f''(k^*) & c^* f''(k^*) \\ -1 & \rho \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}.$$

The eigenvalues satisfy the characteristic polynomial,

$$\lambda^2 + \left(\beta c^* f''(k^*) - \rho\right) \lambda + \beta c^* f''(k^*) = 0.$$

The roots are given by

$$\lambda_{\pm} = \frac{1}{2}\left[-\left(\beta c^{*} f''(k^{*}) - \rho\right) \pm \sqrt{\left(\beta c^{*} f''(k^{*}) - \rho\right)^{2} - 4\beta c^{*} f''(k^{*})}\right],$$

one of which is $\lambda_{-} < 0$ and the other is $\lambda_{+} > -\left(\beta c^{*} f''(k^{*}) - \rho\right)$.

(e) Ans: The expressions for the eigenvectors are

$$\begin{bmatrix} -\beta c^{*} f''(k^{*}) & c^{*} f''(k^{*}) \\ -1 & \rho \end{bmatrix} \nu_{\pm} = \lambda_{\pm} \nu_{\pm}.$$

The eigenvector associated with λ_{-}, ν_{-}, has slope given by

$$\frac{c - c^{*}}{k - k^{*}} = \rho - \lambda_{-} > 0$$

and the eigenvector associated with λ_{+}, ν_{+}, has slope given by

$$\frac{c - c^{*}}{k - k^{*}} = \rho - \lambda_{+} < \beta c^{*} f''(k^{*}) < 0.$$

The linearized system is saddle-path stable since one of the two eigenvalues is positive while the other is negative.

2. (a) Ans: The steady state is given by the solution to the two conditions

$$1 = \beta\left(1 + A f'(k^{*}) - \delta\right)$$

and

$$c^{*} = A f(k^{*}) - \delta k^{*}.$$

Differentiating with respect to A,

$$dk^{*} = -\frac{f'(k^{*})}{A f''(k^{*})} dA$$

and

$$\begin{aligned} dc^{*} &= \left(A f'(k^{*}) - \delta\right) dk^{*} + f(k^{*})\, dA \\ &= \rho\, dk^{*} + f(k^{*})\, dA \\ &= \left(f(k^{*}) - \rho \frac{f'(k^{*})}{A f''(k^{*})}\right) dA, \end{aligned}$$

where the last step substituted the first derivative into the second. Both k^{*} and c^{*} increase with A.

(b) Ans: The linearized dynamic equations are

$$c_{t+1} - c_{t} = c^{*} A f''(k^{*})(k_{t} - k^{*}) - \beta c^{*} A f''(k^{*})(c_{t} - c^{*})$$

and

$$k_{t+1} - k_{t} = \rho(k_{t} - k^{*}) - (c_{t} - c^{*}).$$

In matrix form,

$$\begin{bmatrix} c_{t+1} - c_{t} \\ k_{t+1} - k_{t} \end{bmatrix} = \begin{bmatrix} -\beta c^{*} A f''(k^{*}) & c^{*} A f''(k^{*}) \\ -1 & \rho \end{bmatrix} \begin{bmatrix} c_{t} - c^{*} \\ k_{t} - k^{*} \end{bmatrix}.$$

The eigenvalues solve the characteristic polynomial,

$$\lambda^{2} + \left(\beta c^{*} A f''(k^{*}) - \rho\right)\lambda + \beta c^{*} A f''(k^{*}) = 0$$

and the eigenvectors solve

$$\left[\begin{array}{ll} -\beta c^* A f''(k^*) & c^* A f''(k^*) \\ -1 & \rho \end{array}\right] \nu_\pm = \lambda_\pm \nu_\pm .$$

The eigenvector associated with λ_-, ν_-, has slope given by

$$\frac{c-c^*}{k-k^*} = \rho - \lambda_- = \frac{-c^* A f''(k^*)}{\lambda_- - \beta c^* A f''(k^*)} > 0.$$

You can demonstrate that $\frac{d\lambda_-}{dA} < 0$ (to do so, you use the result that $1+\lambda_- > 0$ for strictly concave $f(k)$ and $\rho > 0$) so that the stable eigenvector slope increases with A .

(c) Ans: Your phase diagram will illustrate the upward shift in the $\Delta k_{t+1} = 0$ locus and the outward shift in the $\Delta c_{t+1} = 0$ locus.

(d) Ans: The solution for c_0 will lie on the stable saddle path about the new steady state. At time 0, the steady state moves to (c^*, k^*) from $(\overline{c}_0, k_0)$ and consumption moves immediately from $\overline{c}_0$ to c_0.

The change in the steady state is

$$\left[\begin{array}{l} c^* - \overline{c}_0 \\ k^* - k_0 \end{array}\right] = \left[\begin{array}{l} \left(f(k^*) - \rho \frac{f'(k^*)}{A f''(k^*)}\right) \\ -\frac{f'(k^*)}{A f''(k^*)} \end{array}\right] \Delta A$$

so that we can write

$$\begin{aligned} c^* - \overline{c}_0 &= -\left(f(k^*) - \rho \frac{f'(k^*)}{A f''(k^*)}\right) \frac{A f''(k^*)}{f'(k^*)} (k^* - k_0) \\ &= \left(\rho - \frac{A f''(k^*)}{f'(k^*)} f(k^*)\right) (k^* - k_0) . \end{aligned}$$

This is equivalent to

$$\overline{c}_0 - c^* = \left(\rho - \frac{A f''(k^*)}{f'(k^*)} f(k^*)\right) (k_0 - k^*)$$

The slope of the new saddle path is given by $\rho - \lambda_-$, so that

$$c_0 - c^* = (\rho - \lambda_-)(k_0 - k^*) .$$

The change in consumption at $t = 0$ is

$$c_0 - \overline{c}_0 = \left(-\lambda_- + \frac{A f''(k^*)}{f'(k^*)} f(k^*)\right) (k_0 - k^*) ,$$

where we know that $k_0 < k^*$ for $\Delta A > 0$. The term $\left(-\lambda_- + \frac{A f''(k^*)}{f'(k^*)} f(k^*)\right)$ cannot be signed without a specific form for the function $f(k)$.

3. (a) Ans: In this part, you solve backwards for k_t from k_0. The solution is

$$k_t = (1+A)^t k_0 - \sum_{s=0}^{t-1} (1+A)^{t-s-1} c_s$$

(b) Ans: This part solves the difference equation forwards. You solve for $(1+A)^{-T} k_{t+T}$ for any $T > 0$ by multiplying the equation of motion,

$$k_{t+1} = (1+A) k_t - c_t$$

by $(1+A)^{-t}$ for each t and summing. The solution is

$$k_t = \left(\frac{1}{1+A}\right)^T k_{t+T} + \sum_{s=t}^{t+T-1} \left(\frac{1}{1+A}\right)^{s-t+1} c_s.$$

(c) Ans: Letting $T \to \infty$, we get

$$k_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+A}\right)^{s-t+1} c_s + \lim_{T\to\infty} \left(\frac{1}{1+A}\right)^T k_{t+T}.$$

This is a constraint on the planner's consumption plan and permanent holding of capital. The intertemporal budget constraint can be written as

$$k_t \geq \sum_{s=t}^{\infty} \left(\frac{1}{1+A}\right)^{s-t+1} c_s$$

by imposing the solvency condition,

$$\lim_{T\to\infty} \left(\frac{1}{1+A}\right)^T k_{t+T} \geq 0.$$

Beginning at time $t = 0$, the budget constraint is

$$k_0 \geq \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t$$

(d) Ans: For this part, we just maximize the utility function $U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t)$ with respect to the sequence $\{c_t\}_{t=0}^{\infty}$ subject to the budget set given by

$$c_t \geq 0, \text{ for all } t \geq 0 \quad \text{and} \quad k_0 \geq \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t.$$

Writing out the Lagrangian,

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left[k_0 - \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t\right],$$

we get the first-order conditions,

$$\beta^t u'(c_t) = \left(\frac{1}{1+A}\right)^{t+1} \lambda \quad \text{for all } t \geq 0$$

and

$$k_0 = \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t \quad \text{for } \lambda > 0.$$

Using the first-order conditions, notice that

$$u'(c_t) = \beta u'(c_{t+1})(1+A) \quad \text{for all } t \geq 0$$

and

$$u'(c_0) = \frac{1}{1+A}\lambda.$$

Using the budget constraint for $\lambda > 0$ (that is, we assume that $u'(c) > 0$ for all $\dot{c} > 0$), $k_0 = \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t$, these conditions solve for $(c_0, c_{2,\ldots.})$ and λ.

4. (a) Ans: Write the Lagrangian,

$$L = \sum_{t=0}^{\infty} \left[\beta^t u(c_t) + \lambda_t (k_t + Ak_t - c_t - k_{t+1})\right]$$

and derive the necessary conditions,

$$\frac{\partial L}{\partial c_t} = 0 \quad \Rightarrow \quad \beta^t u'(c_t) = \lambda_t,$$
$$\frac{\partial L}{\partial k_{t+1}} = 0 \quad \Rightarrow \quad \lambda_{t+1}(1+A) = \lambda_t,$$
$$\frac{\partial L}{\partial \lambda_t} = 0 \quad \Rightarrow \quad k_{t+1} = (1+A)k_t - c_t,$$

and

$$\lim_{T\to\infty} \lambda_T k_{T+1} = 0 \quad \text{and} \quad \lim_{T\to\infty} k_{T+1} \geq 0.$$

These can be simplified to

$$u'(c_t) = \beta u'(c_{t+1})(1+A)$$
$$k_{t+1} = (1+A)k_t - c_t,$$

and

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \quad \text{and} \quad \lim_{T\to\infty} k_{T+1} \geq 0.$$

Note that this assumed an interior solution. A sufficient condition is that $\lim_{c\to 0} u'(c) = \infty$ and $u''(c) < 0$ for all $c > 0$.

(b) Let $u(c) = \log c$. The Euler condition becomes

$$\frac{c_{t+1}}{c_t} = \beta(1+A),$$

which can be iterated backward to

$$c_t = \beta^t (1+A)^t c_0.$$

Substituting this into the intertemporal budget constraint, we get

$$\begin{aligned}
k_0 &= \sum_{s=0}^{\infty} \left(\frac{1}{1+A}\right)^{s+1} c_s + \lim_{t\to\infty} \left(\frac{1}{1+A}\right)^t k_t \\
&= \sum_{s=0}^{\infty} \left(\frac{1}{1+A}\right)^{s+1} \beta^s (1+A)^s c_0 + \lim_{t\to\infty} \left(\frac{1}{1+A}\right)^t k_t \\
&= \frac{1}{1+A}\frac{1}{1-\beta} c_0 + \lim_{t\to\infty} \left(\frac{1}{1+A}\right)^t k_t.
\end{aligned}$$

Now, substitute $u'(c_t) = \frac{1}{c_t} = \frac{1}{c_0}\beta^{-t}(1+A)^{-t}$ into the transversality condition,

$$\begin{aligned}
\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} &= \lim_{t\to\infty} \beta^t \frac{1}{c_0} \beta^{-t} (1+A)^{-t} k_{t+1} = \frac{1}{c_0} \lim_{t\to\infty} \left(\frac{1}{1+A}\right)^t k_{t+1} \\
&= (1+A)\frac{1}{c_0} \lim_{t\to\infty} \left(\frac{1}{1+A}\right)^{t+1} k_{t+1} = 0.
\end{aligned}$$

Thus,

$$k_0 = \frac{1}{1+A}\frac{1}{1-\beta} c_0$$

so that

$$c_0 = (1 - \beta)(1 + A) k_0.$$

You can generalize this immediately to

$$c_t = (1 - \beta)(1 + A) k_t, \qquad \text{for all } t \geq 0.$$

(c) Ans: You just need to explain that the transversality condition, $\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0$, led to $\lim_{t \to \infty} \left(\frac{1}{1+A}\right)^t k_t = 0$. That tells us that the constraint that $\lim_{T \to \infty} k_{T+1} \geq 0$ implies that the $\lim_{t \to \infty} \left(\frac{1}{1+A}\right)^t k_t \geq 0$. You should notice that $\lim_{t \to \infty} \left(\frac{1}{1+A}\right)^t k_t \geq 0$ is not sufficient to imply that $\lim_{T \to \infty} k_{T+1} \geq 0$. The transversality condition tells us that $\lim_{t \to \infty} \left(\frac{1}{1+A}\right)^t k_t = 0$ so that the intertemporal budget constraint,

$$k_0 \geq \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t$$

holds with equality as

$$k_0 = \sum_{t=0}^{\infty} \left(\frac{1}{1+A}\right)^{t+1} c_t.$$