Economics 205A
Fall 2016
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Midterm Exam with Sample Answers

1. Use the basic representative agent economy to answer this question. The household's utility function is

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t),$$

the production function is $y_t = f(k_t)$ and there are quadratic costs of investment.

a) Define a competitive equilibrium for this decentralized economy assuming firms purchase capital and households own firms.

Ans: A competitive equilibrium is an allocation, $\{c_t, k_{t+1}\}$, and prices, $\{w_t, r_{t+1}\}_{t=0}^{\infty}$, such that

i) households maximize their utilities over their budget sets given these prices;

ii) firms maximize their value given these prices; and

iii) all markets clear (these are the markets for goods, factors and assets).

b) Write out the problems and conditions needed to determine the competitive equilibrium.

Ans: i)The representative household's problem is

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the budget identity,

$$a_{t+1} = (1 + r_t) a_t + w_t - c_t,$$

and the solvency constraint,

$$\lim_{T\to\infty} R_{0,T}^{-1} a_{T+1} \geq 0,$$

given initial assets a_0. (Assume that labor endowment equals 1.) The notation $R_{0,t}^{-1} = (1 + r_0) \prod_{s=0}^{t} \left(\frac{1}{1+r_s}\right)$ is used.

ii) The representative firm's problem is

$$V(k_0 - b_0) = \max_{\{n_t, i_t, x_t, k_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} R_{0,t}^{-1} \left(n_t f\left(\frac{k_t}{n_t}\right) - w_t n_t - i_t + x_t \right)$$

subject to

$$k_{t+1} = (1 - \delta) k_t + i_t$$

$$b_{t+1} = (1 + r_t) b_t + x_t$$

and the inequalities $k_t \geq 0$, $n_t \geq 0$, and the solvency condition,

$$\lim_{T\to\infty} R_{0,T}^{-1} b_{T+1} \leq 0.$$

This can be written more compactly as

$$V(k_0 - b_0) = \max_{\{n_t, k_t, b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} R_{0,t}^{-1} \left(n_t f\left(\frac{k_t}{n_t}\right) - w_t n_t + (1-\delta) k_t - k_{t+1} - (1+r_t) b_t + b_{t+1} \right)$$

subject to the inequalities on capital and labor inputs and the solvency condition. The initial condition for the firm is that $k_0 - b_0 = 0$ (k_0 and b_0 are choices for the firm, but $k_0 - b_0$ is not. Here, n_t denotes the firm's demand for labor.

iii) The equilibrium conditions are the labor market clearing condition,

$$n_t^d = n_t^s = 1$$

the goods market clearing condition,

$$f(k_t) = c_t + k_{t+1} - (1-\delta) k_t$$

and the asset market equilibrium condition,

$$a_t = b_t + V(k_t - b_t)$$

Thus, $a_0 = b_0 + V(k_0 - b_0)$. Constant returns to scale implies that $V(k_t - b_t) = 0$, so that $a_t = b_t = k_t$.

c) Demonstrate that the competitive equilibrium allocation is a command optimum.

Ans: The necessary conditions for household optimum are the Euler condition,

$$u'(c_t) = \beta (1 + r_{t+1}) u'(c_{t+1}),$$

budget identity,

$$a_{t+1} = (1 + r_t) a_t + w_t - c_t$$

the transversality condition,

$$\lim_{t\to\infty} \beta^t u'(c_{t+1}) a_{t+1} = 0$$

and the initial condition a_0.

The necessary conditions for the firm give us the factor market clearing conditions (incorporating labor market equilibrium, $n_t = 1$),

$$f(k_t) - k_t f'(k_t) = w_t$$

$$f'(k_t) = r_t + \delta.$$

Together, these verify that

$$n_t f\left(\frac{k_t}{n_t}\right) - w_t n_t + (1-\delta) k_t - k_{t+1} - (1+r_t) b_t + b_{t+1} = 0$$

so that $b_t = k_t$ and $V(k_t - b_t) = 0$ for all $t \geq 0$.

Substituting equilibrium conditions into the household budget identity, Euler condition, transversality condition, and initial condition, gives us the resource identity,

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t,$$

the Euler condition as

$$u'(c_t) = (1 + f'(k_{t+1}) - \delta)\beta u'(c_{t+1}),$$

the transversality condition,

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0$$

and the initial condition, $a_0 = k_0$.

d) Suppose that $A_0 = A$ and that the economy is in steady state at $t = 0$. At $t = 0$, everyone learns that A will permanently rise to $A + dA$ at $t = 1$. Explain or show how to determine the response of consumption and investment at $t = 0$ to this anticipated productivity increase

Ans: The best way to do this is the most direct - find the command optimum.

At $t = 0$, the economy begins in the steady state, $(\bar{c}, \bar{k})$, determined by

$$\bar{c} = Af(\bar{k}) + (1-\delta)\bar{k}$$

and

$$\beta(1 + Af'(\bar{k}) - \delta) = 1.$$

At $t = 1$, the new steady state, (c^*, k^*), is determined by

$$c^* = (A + dA) f(k^*) + (1-\delta) k^*$$

and

$$\beta(1 + (A + dA) f'(k^*) - \delta) = 0.$$

From date $t = 1$ onward, consumption and capital stock are determined by the command optimum given the technology, $y = (A + dA) f(k)$. The solution for (c_1, k_1) will be on the saddle path converging to (c^*, k^*).

The saddle path is given by

$$\frac{c_t - c^*}{k_t - k^*} = \rho - \lambda_- \ .$$

where λ_- is the negative eigenvalue of the matrix for the linearized dynamics,

$$M = \begin{bmatrix} \beta \frac{u'(c^*)}{u''(c^*)} Af''(k^*) & -\frac{u'(c^*)}{u''(c^*)} Af''(k^*) \\ -1 & \rho \end{bmatrix}$$

(You were not expected to write this matrix out.) Thus,

$$(c_1 - c^*) = (\rho - \lambda_-)(k_1 - k^*) \tag{1}$$

To find c_0, c_1 and k_1, we need to use the resource identity to get

$$(c_0 - c^*) + (k_1 - k^*) = 0 \tag{2}$$

(remember the production function at $t = 0$ is $y = Af(k)$) and the Euler condition to get

$$u'(c_0) = \beta(1 + (A + dA) f'(k_1) - \delta) u'(c_1) \tag{3}$$

Together, equations 1,2, and 3 provide the solutions for consumption, c_0, and investment, $k_1 - \overline{k}$ at $t = 0$.

2. A household seeks to maximize utility,

$$U_0 = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \log \ell_t \right)$$

for $\sigma > 0$. The household can borrow or lend freely at a constant rate of interest rate r and receives a wage w_t for each day worked. The endowment of leisure, ℓ , each day equals one.

a) Write out the optimization problem for the household. Identify the conditions or constraints you introduce

Ans: The optimization problem is

$$\max_{\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\frac{c_t^{1-\sigma}}{1-\sigma} c_t + \log(1 - n_t))$$

subject to the budget identity,

$$a_{t+1} = (1 + r) a_t + w_t n_t - c_t,$$

the solvency constraint,

$$\lim_{t \to \infty} \left(\frac{1}{1+r} \right)^t a_{t+1} \geq 0,$$

the inequalitity constraints for labor supply,

$$1 \geq n_t \geq 0,$$

given initial assets a_0.(You can write this using ℓ_t just as well.)

b) Find the necessary conditions for an optimum. Label what these are.

Ans: The necessary conditions are the Euler condition,

$$c_t^{-\sigma} = \beta(1 + r) c_{t+1}^{-\sigma}$$

the first-order condition between the consumption and labor supply (you can call this the labor-leisure choice),

$$\frac{1}{1 - n_t} = w_t c_t^{-\sigma},$$

the budget identity,

$$a_{t+1} = (1 + r) a_t + w_t n_t - c_t,$$

the transversality condition,

$$\lim_{t \to \infty} \beta^t c_t^{-\sigma} a_{t+1} = 0$$

and the solvency condition, $\lim_{t \to \infty} \left(\frac{1}{1+r} \right)^t a_{t+1} \geq 0$, plus the initial condition, a_0. Using the Euler condition, the transversality condition implies that the solvency constraint holds with equality, $\lim_{t \to \infty} \left(\frac{1}{1+r} \right)^t a_{t+1} = 0$

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c) Let $r = \rho$, and show how you can use the necessary conditions to solve for household consumption and labor supply as functions of permanent income.

Ans: Integrating the budget identity using the solvency constraint and transversality condition, we have

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}c_t = (1+r)\,a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}w_t n_t.$$

Using the Euler condition and $\frac{1}{1+r} = \beta$, we can substitute for consumption using $c_t = c_0$ to get

$$\frac{1+r}{r}c_0 = (1+r)\,a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}w_t n_t.$$

We can rearrange this to write consumption as a function of permanent income,

$$c_0 = ra_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}w_t n_t,$$

where $\frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}w_t n_t$ is permanent labor income. More generally,

$$c_t = ra_t + \frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}w_s n_s \equiv ra_t + \widetilde{w_t n_t}.$$

The consumption-labor supply equilibrium condition gives us

$$w_s\left(1-n_s\right) = c_s^{\sigma}$$

which we substitute to get the implicit solution for c_t,

$$c_t + c_t^{\sigma} = ra_t + \frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}w_s = ra_t + \widetilde{w}_t$$

We can similarly express labor supply implicitly as a function of permanent income,

$$\left(w_t(1-n_t)\right)^{\frac{1}{\sigma}} + w_t\left(1-n_t\right) = ra_t + \widetilde{w}_t.$$

d) Suppose that the household anticipates the wage rate to equal a constant w forever. At $t = 0$, it is surprised and receives a higher wage equal to $w + dw$ for just date 0 . Show how this unanticipated temporary increase affects consumption and labor supply at $t = 0$ and for $t \geq 1$. How does it affect household savings?

Ans: The unanticipated increase in the wage rate will affect the substitution between consumption and leisure at both $t = 0$ and for $t > 0$ as well as the levels of each for all $t \geq 0$. The change in consumption, dc_t, is the same for $t \geq 0$. Beginning with the equation we derived for c_t, we can write

$$c_t + w\ell_t = ra_t + \widetilde{w}_t$$

where the left-hand side is household expenditure on consumption of goods and leisure. The change in expenditure equals the change in the permanent income from the household's endowment,

$$d\left(c_0 + w\ell_0\right) = \frac{r}{1+r} dw$$

Using the first-order condition for the consumption-leisure choice, this can be rewritten to show the change in c_t,

$$dc_t = dc_0 = \frac{1}{1+\sigma c_0^{\sigma-1}} \frac{r}{1+r} dw.$$

Labor income at $t=0$ rises by

$$d\left(wn\right)_0 = dc_0 + \frac{1}{1+r} dw,$$

labor supply at time $t=0$ satisfies,

$$\frac{dn_0}{1-n_0} = \frac{dw}{w} - \sigma \frac{dc_0}{c_0}.$$

and labor supply for $t>0$ changes as

$$\frac{dn_t}{1-n_0} = -\sigma \frac{dc_0}{c_0}.$$

We see that labor supply decreases when wages return to equal w as consumption of goods and leisure rise permanently, but the sign of the change in labor supply at time 0 depends on the elasticity of substitution σ and financial wealth a_0.

Savings is given by

$$\Delta a_1 = ra_0 + wn_0 - c_0 + d\left(wn_0\right) - dc_0 = \frac{1}{1+r} dw$$

because $ra_0 + wn_0 - c_0 = 0$ for $r = \rho$.