Economics 205C
Spring 2017

Problem Set 1: Answers

Group work is fine, but if you work with a group, put the names of all members of the group on the assignment you turn in.

1. The Euler condition for optimal consumption in a basic NK model takes the form

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(\frac{1+i_t}{1+\pi_{t+1}} \right) C_{t+1}^{-\sigma}, \tag{1}$$

where the utility of consumption is $C_t^{1-\sigma}/(1-\sigma)$.

(a) Linearize this condition around a zero-inflation steady state. *Use the following:* $X_t \equiv X(1+\hat{x}_t)$ *where* X *is the steady-state value of* X_t *and* $\hat{x}_t$ *is the percent deviation of* X_t *around* X*. To first order,* $X_t^a \approx X(1+a\hat{x}_t)$*. Also,* $(1+\hat{x}_t)(1+\hat{y}_t) = 1+\hat{x}_t+\hat{y}_t+\hat{x}_t\hat{y}_t \approx 1+\hat{x}_t+\hat{y}_t$ *to first order. Since* i *and* π *are already in percent terms and* $\beta = 1/(1+r)$,

$$\beta \left(\frac{1+i_t}{1+\pi_{t+1}} \right) \approx 1 + i_t - \pi_{t+1} - r.$$

The Euler condition is approximated as

$$C^{-\sigma}(1-\sigma\hat{c}_t) \approx \mathrm{E}_t \left(1+i_t-\pi_{t+1}-r\right) C^{-\sigma}(1-\sigma\hat{c}_{t+1})$$

$$\begin{aligned}(1-\sigma\hat{c}_t) &= \mathrm{E}_t\left(1+i_t-\pi_{t+1}-r\right)(1-\sigma\hat{c}_{t+1}) \\ &\approx \mathrm{E}_t\left(1+i_t-\pi_{t+1}-r-\sigma\hat{c}_{t+1}\right)\end{aligned}$$

or

$$-\sigma\hat{c}_t = i_t - \mathrm{E}_t\pi_{t+1} - r - \sigma\mathrm{E}_t\hat{c}_{t+1}$$

yielding

$$\hat{c}_t = \mathrm{E}_t\hat{c}_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t\pi_{t+1} - r\right).$$

(b) Now assume households display habit persistence, with the utility of consumption given by $(C_t - hC_{t-1})^{1-\sigma}/(1-\sigma)$, $h > 0$. Derive the Euler condition for the optimal intertemporal allocation of consumption in the following two cases:

i. External habit persistence – the household takes C_{t-1} as given by aggregate consumption when it chooses C_t. *The marginal utility of consumption is*

$$(C_t - hC_{t-1})^{-\sigma}$$

so the Euler equation becomes

$$(C_t - hC_{t-1})^{-\sigma} = \beta\mathrm{E}_t\left(\frac{1+i_t}{1+\pi_{t+1}}\right)(C_{t+1} - hC_t)^{-\sigma}.$$

ii. Internal habit persistence – the household treats C_{t-1} as its own past consumption when it chooses C_t. *The first-order condition for the optimal consumption choice is*

$$\lambda_t = (C_t - hC_{t-1})^{-\sigma} - \beta \mathrm{E}_t (C_{t+1} - hC_t)^{-\sigma}$$

so the Euler equation becomes

$$(C_t - hC_{t-1})^{-\sigma} - \beta \mathrm{E}_t (C_{t+1} - hC_t)^{-\sigma} = \beta \mathrm{E}_t \left(\frac{1+i_t}{1+\pi_{t+1}} \right) \left[(C_{t+1} - hC_t)^{-\sigma} - \beta \mathrm{E}_t (C_{t+2} - hC_{t+1})^{-\sigma} \right].$$

(c) For the case of part b.i, derived the linearized Euler condition. How does it compare to (1)? Explain. *Follow the earlier steps using the result that*

$$(C_t - hC_{t-1})^{-\sigma} \approx C^{-\sigma} (1-h)^{-\sigma} (1 - \sigma \hat{x}_t)$$

where $\hat{x}_t$ is defined such that

$$\begin{aligned} \hat{x}_t &= \frac{C_t - hC_{t-1} - C(1-h)}{C(1-h)} \\ &= \frac{(C_t - C) - h(C_{t-1} - C)}{C(1-h)} \\ &= \frac{1}{1-h}\hat{c}_t - \left(\frac{h}{1-h} \right) \hat{c}_{t-1}. \end{aligned}$$

Hence,

$$(C_t - hC_{t-1})^{-\sigma} \approx (1-h)^{-\sigma} C^{-\sigma} \left[1 - \frac{\sigma}{1-h}\hat{c}_t + \sigma \left(\frac{h}{1-h} \right) \hat{c}_{t-1} \right]$$

and the Euler condition is

$$\left[1 - \frac{\sigma}{1-h}\hat{c}_t + \sigma \left(\frac{h}{1-h} \right) \hat{c}_{t-1} \right] = \beta \mathrm{E}_t \left(\frac{1+i_t}{1+\pi_{t+1}} \right) \left[1 - \frac{\sigma}{1-h}\hat{c}_{t+1} + \sigma \left(\frac{h}{1-h} \right) \hat{c}_t \right]$$

$$\left[1 - \frac{\sigma}{1-h}\hat{c}_t + \sigma \left(\frac{h}{1-h} \right) \hat{c}_{t-1} \right] = \mathrm{E}_t \left(1 + i_t - \pi_{t+1} - r \right) \left[1 - \frac{\sigma}{1-h}\hat{c}_{t+1} + \sigma \left(\frac{h}{1-h} \right) \hat{c}_t \right]$$

$$-\frac{\sigma}{1-h}\hat{c}_t + \sigma \left(\frac{h}{1-h} \right) \hat{c}_{t-1} \approx i_t - \mathrm{E}_t \pi_{t+1} - r - \frac{\sigma}{1-h}\mathrm{E}_t \hat{c}_{t+1} + \sigma \left(\frac{h}{1-h} \right) \hat{c}_t$$

$$\sigma \left(\frac{1+h}{1-h} \right) \hat{c}_t \approx -\left(i_t - \mathrm{E}_t \pi_{t+1} - r \right) + \sigma \left(\frac{h}{1-h} \right) \hat{c}_{t-1} + \frac{\sigma}{1-h}\mathrm{E}_t \hat{c}_{t+1}$$

$$\hat{c}_t = \left(\frac{1}{1+h} \right) \mathrm{E}_t \hat{c}_{t+1} + \left(\frac{h}{1+h} \right) \hat{c}_{t-1} - \left(\frac{1}{\sigma} \right) \left(\frac{1-h}{1+h} \right) \left(i_t - \mathrm{E}_t \pi_{t+1} - r \right).$$

The marginal utility of depends on past consumption, so both lagged consumption and expected future consumption appear in the linearsized Euler equation. The larger is the value of h, the more past consumption influences current optimal consumption.

2. Consider the following basic NK model:

$$x_t = \mathrm{E}_t x_{t+1} - \sigma^{-1} \left(i_t - \mathrm{E}_t \pi_{t+1} - r_t \right)$$
$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + u_t$$
$$u_t = \rho u_{t-1} + v_t,$$

where x is the output gap, i is the nominal interest rate, π is the inflation rate, r is the equilibrium real interest rate when prices are flexible, and u is a cost shock. Assume r and v are exogenous stochastic iid disturbances. Assume monetary policy is given by

$$i_t = \phi \pi_t.$$

(a) Write the model in the form

$$\begin{bmatrix} \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t \pi_{t+1} \end{bmatrix} = M \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + N \begin{bmatrix} r_t \\ u_t \end{bmatrix}.$$

Give explicit expressions for the 2×2 matrices M and N. *First rewrite model using policy rule as*

$$x_t = \mathrm{E}_t x_{t+1} - \sigma^{-1} \left(\phi \pi_t - \mathrm{E}_t \pi_{t+1} - r_t \right)$$
$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + u_t$$

Then write it as

$$\mathrm{E}_t x_{t+1} + \sigma^{-1} \mathrm{E}_t \pi_{t+1} = x_t + \sigma^{-1} \phi \pi_t - \sigma^{-1} r_t$$
$$\beta \mathrm{E}_t \pi_{t+1} = \pi_t - \kappa x_t - u_t$$

or

$$\begin{aligned} \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t \pi_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & \sigma^{-1}\phi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \\ &\quad + \begin{bmatrix} -\sigma^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r_t \\ u_t \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t \pi_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix}^{-1} \begin{bmatrix} 1 & \sigma^{-1}\phi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix}^{-1} \begin{bmatrix} -\sigma^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r_t \\ u_t \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t \pi_{t+1} \end{bmatrix} &= \begin{bmatrix} \frac{\kappa}{\sigma\beta} + 1 & \frac{1}{\sigma}\phi - \frac{1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{1}{\sigma} & \frac{1}{\sigma\beta} \\ 0 & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} r_t \\ u_t \end{bmatrix}. \end{aligned}$$

Hence,

$$M = \begin{bmatrix} \frac{\kappa + \sigma\beta}{\sigma\beta} & \frac{\beta\phi - 1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

$$N = \begin{bmatrix} -\frac{1}{\sigma} & \frac{1}{\sigma\beta} \\ 0 & -\frac{1}{\beta} \end{bmatrix}.$$

(b) The Blanchard-Kahn conditions for a locally unique rational expectations solution require that both eigenvalues of M lie outside the unit circle. Assume $\sigma = 1$, $\eta = 1$, $\beta = 0.99$, $\rho = 0.9$, and $\kappa = (1-\omega)(1-\beta\omega)(\sigma+\eta)/\omega$, where ω is the Calvo parameter. Set $\omega = 0.75$. Plot the absolute values of the two eigenvalues of M as functions of ϕ. Verify that the Taylor principle $\phi > 1$ is necessary if the Blanchard-Kahn condition is to be satisfied. See *nk_ eigenvalues_ plots.m*

(c) Set $\phi = 1.5$. Solve the model numerically using dynare and plot the responses of the output gap and inflation to a positive realization of v_t. See *nk_ ps_ 1_ 2b_ and_ c.dyn.*

(d) Repeat part (c) with $\phi = 1.1$. Explain why the responses differ from those you obtained in part (c). *With the smaller value of ϕ, the central bank is responding less strongly to inflation, so inflation moves more in response to the shock.*

3. Repeat part (c) of question 3 using the linearized Euler condition you obtained in part 2.c above if $h = 0.8$. Explain how the responses differ from those obtained in 3.c. *See nk_ ps_ 1_ q3.dyn. Habit persistence adds some endogenous dynamics to the model. this can be seen most clearly if the structural shocks r and u are assumed to be serially uncorrelated. In the absence of habit persistence, consumption responds immediately to the shock but then returns to its steady-state value in the next period if the shock is serially uncorrelated. In a contrast, consumption adjusts in response to lagged consumption when $h > 0$, so the movement of consumption in teh period the shock hits continues to affect consumption in subsequent periods.*

4. Suppose the central bank cares about inflation variability, output gap, variablility and interest rate variability. The objective of the central bank is to minimize

$$\left(\frac{1}{2}\right) \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left[\pi_{t+i}^2 + \lambda_x x_{t+i}^2 + \lambda_i \left(i_{t+i} - i^*\right)^2\right].$$

The structure of the economy is given by

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t \tag{2}$$

$$x_t = E_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t\right), \tag{3}$$

where e and r are exogenous stochastic shocks. Let ψ_t denote the Lagrangian multiplier on constraint (2) and let θ_t be the multipler on (3). *Under discretion, the central bank takes expectations as given, so its decision problem can be written as*

$$\min_{\pi_t, x_t, i_t} \left(\frac{1}{2}\right)\left[\pi_t^2 + \lambda_x x_t^2 + \lambda_i \left(i_t - i^*\right)^2\right] + \psi_t \left[\pi_t - \beta \mathrm{E}_t \pi_{t+1} - \kappa x_t - e_t\right]$$
$$+\theta_t \left[x_t - \mathrm{E}_t x_{t+1} + \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t\right)\right].$$

The first order conditions are

$$\pi_t + \psi_t = 0$$

$$\lambda_x x_t - \kappa\psi_t + \theta_t = 0$$

and

$$\lambda_i \left(i_t - i^*\right) + \left(\frac{1}{\sigma}\right)\theta_t = 0.$$

(a) Show that θ is nonzero if $\lambda_i > 0$. Explain the economics behind this result. *From this last condition, if $\lambda_i = 0$, then $\theta_t = 0$ and the IS relationship does not impose a constraint on the central bank's policy choice. Because, when $\lambda_i = 0$, the central bank does not care about interest rate volatility, it can always adjust i_t to offset any shocks arising from the IS relationship, preventing these from affecting the things it does care about – inflation and the output gap.*

(b) Derive the first-order conditions for the fully optimal commitment policy. How do these differ from the conditions you found in (a)? *Under optimal commitment, the central bank's decision problem is*

$$\min \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ \frac{1}{2}\left[\pi_{t+i}^2 + \lambda x_{t+i}^2 + \lambda_i \left(i_{t+i} - i^*\right)^2\right] \right.$$
$$+\psi_{t+i}\left(\pi_{t+i} - \beta\pi_{t+1+i} - \kappa x_{t+i} - e_{t+i}\right)$$
$$\left. +\theta_{t+i}\left[x_{t+i} - \mathrm{E}_t x_{t+1+i} + \left(\frac{1}{\sigma}\right)\left(i_{t+i} - \pi_{t+1+i} - r_{t+i}\right)\right]\right\}.$$

The first order conditions are

$$\text{for } \pi_t\text{: } \pi_t + \psi_t = 0 \tag{4}$$

$$\text{for } \pi_{t+i}\text{: } \pi_{t+i} + \psi_{t+i} - \psi_{t+i-1} - \left(\frac{1}{\sigma\beta}\right)\theta_{t+i-1} = 0 \text{ for } i > 0 \tag{5}$$

$$\text{for } x_t\text{: } \lambda_x x_t - \kappa\psi_t + \theta_t = 0. \tag{6}$$

$$\text{for } x_{t+i}\text{: } \lambda_x x_{t+i} - \kappa\psi_{t+i} + \theta_{t+i} - \left(\frac{1}{\beta}\right)\theta_{t+i-1} = 0 \text{ for } i > 0. \tag{7}$$

$$\text{for } i_{t+i}\text{: } \lambda_i \left(i_{t+i} - i^*\right) - \left(\frac{1}{\sigma}\right)\theta_{t+i} = 0 \text{ for } i \geq 0. \tag{8}$$

For $i = 0$, the conditions are the same as in part (a). For $i > 0$, they differ. From the inflation adjustment equation, a central bank, faced with a cost shock, can better stabilize current inflation if it can adjust both the current output gap and the private sector's expectations about future inflation. Or, since the central bank cares about both output gap and inflation stabilization, we can express this by saying that if the cost shock is positive, a given rise ininflation can be achieved with a smaller decline in the output gap if expected future inflation is reduced. Thus, the optimal commitment policy will promise a deflation in period $t+1$ so that $\mathrm{E}_t\pi_{t+1} < 0$. This promise must be fulfilled, so at time $t+1$ the cenral bank's actions reflect promises made at time t, making policy backward looking. Because $\lambda_i > 0$, the central bank cares about interest rate volatility, so the IS relationship becomes a constraint on policy choices ($\theta_t > 0$), and moving i_t to offset movements in r_t may lead to undesired interest rate volatility. Since the current output gap is affected by the expected future output gap through the IS relationship, the plan for x_{t+i}, $i > 0$, must take into account the effect this has on x_{t+i-1}, just as the choice of π_{t+i}, $i > 0$, affects π_{t+i-1}. This accounts for the θ_{t-1} term in (7). It is absent in (6) since x_t can no longer affect x_{t-1}.

(c) Derive the first-order conditions for the optimal commitment policy from a timeless perspective. How do these differ from the conditions you found in (c)? *Under the timeless perspective, policy choices are such that (5), (7), and (8) are staisfied for all* $i \geq 0$. *That is, the special nature of the first period, when the effects on the past can be ignored, is not exploited and instead policy is set to satisfy*

$$\pi_{t+i} + \psi_t - \psi_{t+i-1} - \left(\frac{1}{\sigma\beta}\right)\theta_{t+i-1} = 0,$$

$$\lambda_x x_{t+i} - \kappa\psi_{t+i} + \theta_{t+i} - \left(\frac{1}{\beta}\right)\theta_{t+i-1} = 0,$$

and

$$\lambda_i\left(i_{t+i} - i^*\right) - \left(\frac{1}{\sigma}\right)\theta_{t+i} = 0$$

for all $i \geq 0$. *From the last of these,* $\theta_{t+i} = \sigma\lambda_i\left(i_{t+i} - i^*\right)$. *Using this in the first order condition for* x_{t+i},

$$\lambda_x x_{t+i} - \kappa\psi_{t+i} + \sigma\lambda_i\left(i_{t+i} - i^*\right) - \left(\frac{1}{\beta}\right)\sigma\lambda_i\left(i_{t+i-1} - i^*\right) = 0$$

so

$$\psi_{t+i} = \left(\frac{1}{\kappa}\right)\left[\lambda_x x_{t+i} + \sigma\lambda_i\left(i_{t+i} - i^*\right)\right] - \left(\frac{1}{\kappa}\right)\left(\frac{1}{\beta}\right)\sigma\lambda_i\left(i_{t+i-1} - i^*\right).$$

Using this in the first order condition for inflation yields the optimal targeting rule under the timeless perspective:

$$\begin{aligned}
\pi_{t+i} &= -\left(\psi_t - \psi_{t+i-1}\right) - \left(\frac{1}{\sigma\beta}\right)\theta_{t+i-1} \\
&= -\left(\frac{\lambda_x}{\kappa}\right)\left(x_{t+i} - x_{t+i-1}\right) - \left(\frac{\sigma\lambda_i}{\kappa}\right)\left(i_{t+i} - i_{t+i-1}\right) \\
&\quad + \left(\frac{\sigma\lambda_i}{\beta\kappa}\right)\left(i_{t+i-1} - i_{t+i-2}\right) - \left(\frac{\lambda_i}{\beta}\right)\left(i_{t+i-1} - i^*\right)
\end{aligned}$$

If $\lambda_i = 0$, $\pi_{t+i} = -\left(\lambda_x/\kappa\right)\left(x_{t+i} - x_{t+i-1}\right)$.

5. In no more than three sentences, explain why the optimal commitment policy in a basic NK model introduces inertia by responding to the lagged value of the output gap. *Commitment improves over discretion by making promises about the future that affect expectations in ways that help stabilize current output gap and inflation in the face of shocks. These promises must be respected, so in the future the policy maker will deliver on promises made in the past. This introduces a backward-looking aspect to policy.*