Timeline

Policy fn's

affects my own capital stock

affects mkt price

$c(k_{t-1}, K_{t-1}, z_t)$

$h(k_{t-1}, K_{t-1}, z_t)$

$i(k_{t-1}, K_{t-1}, z_t)$

Q. a general specification. and we're only considering the special case in which we have representative households.

$$\begin{cases} H^d(K_{t-1}, z_t) \\ K^d(K_{t-1}, z_t) \end{cases}$$

- $K_t = g(K_{t-1}, z_t)$

$K_t = (1-\delta) K_{t-1} + i(K_{t-1}, K_{t-1}, z_t)$

If individual households hold the same amount of capital as the aggregate, then their capital next period also coincide with the aggregate.

i.e. individual law of motion of capital = law of motion of aggregate capital

(2) Jan. 12 (Thu.)

RBC model with investment adjustment cost

Law of motion of capital

$$k_t = (1-\delta) k_{t-1} + \left\{ 1 - \frac{\kappa}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right\} i_t$$

$$K_t = (1-\delta) K_{t-1} + \left\{ 1 - \frac{\kappa}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right\} I_t$$

$\kappa > 0$: parameter

when $i_t = i_{t-1}$, $k_t = (1-\delta) k_{t-1} + i_t$

when $\kappa \neq 0$, $k_t = (1-\delta) k_{t-1} + i_t$

Household's Bellman equation

$$V(k, K, z, i_{-1}, I_{-1}) = \max_{c,h,i} \left[u(c,n) + \beta E\left[V(k', K', z', i, I) \right] \right]$$

s.t. $c + i \leq wh + rk$

$$k' = (1-\delta) k + \left\{ 1 - \frac{\kappa}{2} \left(\frac{i}{i_{-1}} - 1 \right)^2 \right\} i$$

Firm's problem

$r(K, z, I_{-1})$ $w(K, z, I_{-1})$

$$\max_{K^d, H^d} \left[z F(K^d, H^d) - r K^d - w H^d \right]$$

ARCE is (i) H's value fn. $V(k, K, z, i_{-1}, I_{-1})$

policy fn.'s $c(\quad " \quad)$

$h(\quad " \quad)$

$i(\quad " \quad)$

(ii) F's policy $K^d(K, z, I_{-1})$

$H^d(K, z, I_{-1})$

(iii) Prices $w(K, z, I_{-1})$

$r(\quad " \quad)$

(iv) law of motion of capital $K' = g(K, z, I_{-1})$ s.t.

①

②

③

④

Consistency condition

$$: g(K, z, I_{-1}) = (1-\delta)\overset{K}{\cancel{k}} + \left\{1 - \frac{\kappa}{2}\left(\frac{i(\overset{K}{\cancel{k}}, K, z, \overset{I_{-1}}{\cancel{i_{-1}}}, I_{-1})}{\underset{I_{-1}}{\cancel{i_{-1}}}} - 1\right)^2\right\} i(\overset{K}{\cancel{k}}, K, z, \overset{I_{-1}}{\cancel{i_{-1}}}, I_{-1})$$

<u>1st Welfare Thm.</u>

C.E. is Pareto optimal.

p.14

① Sequential formulation

<u>Planner's problem</u>

$$\max_{C_t, H_t, I_t} E \sum_{t=0}^{\infty} \beta^t u(C_t, H_t) \quad \text{s.t.} \quad \begin{aligned} C_t + H_t &\leq z_t F(K_{t-1}, H_t) \\ K_t &= (1-\delta)K_{t-1} + I_t \end{aligned}$$

$$\mathcal{L} = E \sum_{t=0}^{\infty} \beta^t \left[u(C_t, H_t) + \lambda_t \left\{ z_t F(K_{t-1}, H_t) - C_t - K_t + (1-\delta)K_{t-1} \right\} \right]$$

FONCs

$[C_t]$ $\lambda_t = U_1(C_t, H_t)$

$[H_t]$ $U_2(C_t, H_t) + \lambda_t z_t F_2(K_{t-1}, H_t) = 0$

$[I_t]$ $\lambda_t = \beta E[\lambda_{t+1}(z_{t+1} F_2(K_t, H_{t+1}) + 1 - \delta)]$

Intratemporal condition

① $-\dfrac{U_2(C_t, H_t)}{U_1(C_t, H_t)} = z_t F_2(\quad)$

MRS = MP_L

Euler eq. (intertemporal condition)

② $\underbrace{U_1(C_t, H_t)}_{\text{MU of consumption}} = \underbrace{\beta E_t[U_1(C_{t+1}, H_{t+1})(z_{t+1} F_2(\quad) + 1 - \delta)]}_{\text{discounted Marginal return on investment}}$

③ $C_t + I_t = Z_t F(K_{t-1}, H_t)$

④ $K_t = (1-\delta) K_{t-1} + I_t$

⑤ $w_t = Z_t F_2 (K_{t-1}, H_t)$

⑥ $r_t = Z_t F_1 (K_{t-1}, H_t)$

(3) Jan. 17 (Tue)

Direct Attack   Sequential formation

Ⓗ's problem

$$\max_{\{c_t, h_t, i_t\}} E \sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \quad \text{s.t.} \quad c_t + i_t \le w_t h_t + r_t k_{t-1}$$
$$k_t = (1-\delta) k_{t-1} + i_t$$

$$\mathcal{L} = E \sum_{t=0}^{\infty} \beta^t u(c_t, h_t) + \lambda_t \{ w_t h_t + r_t k_{t-1} - c_t - k_t + (1-\delta) k_{t-1} \}$$

FONCs: $[c_t]$ $\lambda_t = u_1(c_t, h_t)$

$[h_t]$ $u_2(c_t, h_t) = \lambda_t w_t$

$[k_t]$ $\lambda_t = \beta E_t \{ \lambda_{t+1} (r_{t+1} + (1-\delta)) \}$

$$\Rightarrow \begin{cases} ① \ \boxed{u_2(c_t, h_t) = u_1(c_t, h_t) w_t} \\ ② \ \boxed{u_1(c_t, h_t) = \beta E_t \{ \lambda_{t+1} (r_{t+1} + (1-\delta)) \}} \end{cases}$$

Ⓕ's problem

$$\max_{K_t^d, H_t^d} [z_t F(K_t^d, H_t^d) - w_t H_t^d - r_t K_t^d]$$

$$\Rightarrow \begin{cases} ③ \ \boxed{w_t = z_t F_2(K_t^d, H_t^d)} \\ ④ \ \boxed{r_t = z_t F_1(K_t^d, H_t^d)} \end{cases}$$

Mkt clearing conditions

$k_{t-1} = K_t^d = K_{t-1}$

$h_t = H_t^d = H_t$

$c_t = C_t$

$i_t = I_t$

⑤ $\boxed{C_t + I_t = z_t F(K_{t-1}, H_t)}$

①~⑤ : equil. conditions

Recursive formulation

(H)'s prob.

$$V(k,K,z) = \max_{\{c,h,i\}} \left[u(c,h) + \beta E V(k',K',z') \right]$$

$$\text{s.t.} \quad \left.\begin{array}{l} c + i \le wh + rk \\ k' = (1-\delta)k + i \end{array}\right\} \Rightarrow \begin{array}{l} c + k' - (1-\delta)k \\ = wh + rk. \end{array}$$

$$= \max_{\{h,k'\}} \left[u(wh + rk + (1-\delta)k - k', h) + \beta E V(k',K',z') \right]$$

FONCs

$$[h] \quad 0 = \boxed{u_1(c,h)w + u_2(c,h)} \quad ①$$

$$[k'] \quad 0 = \boxed{-u_1(c,h) + \beta E V_k(k',K',z')} \quad ②$$

Envelope condition :

$$V_k(k,K,z) = u_1(c,h)(r+1-\delta)$$

$$\Rightarrow V_k(k',K',z') = u_1(c',h')(r'+1-\delta)$$

(E)'s prob.

$$\max_{\{K^d,H^d\}} \left[zF(K^d,H^d) - wH^d - rK^d \right]$$

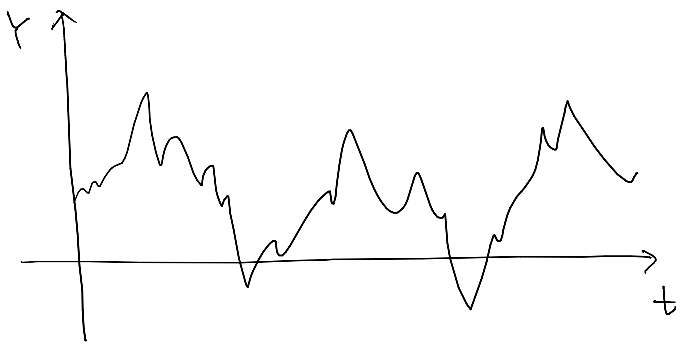
$$\Rightarrow \begin{array}{l} ③\ \boxed{r = zF_1(K^d,H^d)} \\ ④\ \boxed{w = zF_2(K^d,H^d)} \end{array}$$

Mkt clearing conditions.

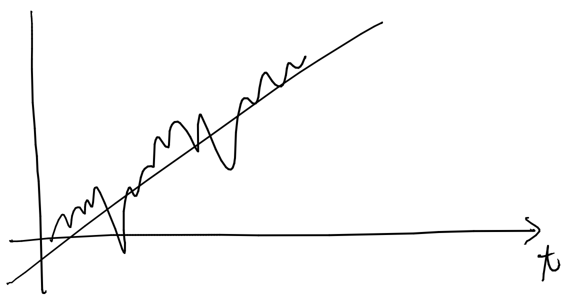
$$\left(\begin{array}{ll} k = K^d = K & c = C \\ h = H^d = H & i = I \end{array} \right. \qquad \boxed{C + I = zF(K,H)} \quad ⑤$$

① ~ ⑤ : Equil. conditions

Introducing Growth.



Model w/ no growth
(RBC we studied so far)



$Y_t = Z_t F(K_{t-1}, X_t H_t)$ where $X_t = \gamma X_{t-1}$ $(\gamma > 0)$

labor-augmenting technological growth factor

Data

- Y, I, C, w, K grow at same rate
- r, H stay constant

(note that pop. is not considered, since in per capita terms)

Eq. conditions

① $U_1(C_t, H_t) = \beta E_t[U_1(C_{t+1}, H_{t+1}) \times \{Z_{t+1} F_1(K_t, X_{t+1}, H_{t+1}) + 1 - \delta\}]$

② $U_2(C_t, H_t) + U_1(C_t, H_t) Z_t F_2(K_{t-1}, X_t, H_t) = 0$

③ $Y_t = Z_t F(K_{t-1}, H_t)$

④ $K_t = (1-\delta)K_{t-1} + I_t$

Functional form

- $F = K_{t-1}^{\alpha} (X_t H_t)^{1-\alpha}$ (C-D fn.)

- $u(C_t, H_t) = \ln C_t - \varphi \frac{H_t^{1+\eta}}{1+\eta}$

φ : scaling factor

η : inverse of Frisch elasticity of labor supply

(more on this when we talk about calibration)

a King-Plosser-Rebelo preference fn. (1988, JME)

↳ general form :

leisure $(= 1-H)$

$u(C, L) = \frac{1}{1-\sigma} C^{1-\sigma} v(L)$

$0 < \sigma < 1$, $v(L)$ increasing, concave

$u(C, L) = \ln C + v(L)$

Why desirable?

KPR pref. satisfies the following restrictions (and hence consistent w/ growth)

① Intertemporal elasticity of substitution (i.e. responsiveness of growth rate of const. to interest rate) is invariant to the scale of constant.

⇒ important because constant growing and hence ratio of discounted marginal utility from the Euler eq. must equal to r, which is not growing

② Income and substitution effects of real wage growth cancels out

⇒ H does not grow.

$\boxed{\text{Eq. Conditions}}$

⇒ ① $\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) \right]$

② $H_t^{\eta} = \frac{1}{C_t} (1-\alpha) \frac{Y_t}{H_t}$

③ $Y_t = Z_t K_{t-1}^{\alpha} (X_t H_t)^{1-\alpha}$

④ $K_t = (1-\delta) K_{t-1} + I_t$

⑤ $C_t + I_t = Y_t$

Consider deflated variables

$$\tilde{Y}_t = \frac{Y_t}{X_t} \quad \longrightarrow \quad Y_t = X_t \tilde{Y}_t$$

$$\tilde{C}_t = \frac{C_t}{X_t} \quad \longrightarrow \quad C_t = X_t \tilde{C}_t$$

$$\tilde{I}_t = \frac{I_t}{X_t} \quad \longrightarrow \quad I_t = X_t \tilde{I}_t$$

$$\tilde{K}_{t-1} = \frac{K_{t-1}}{X_t} \quad \longrightarrow \quad K_{t-1} = X_t \tilde{K}_{t-1}$$

Now, the eq. conditions:

① $$\frac{1}{\cancel{X_t}\tilde{C}_t} \cdot \gamma = \beta E_t\left[\frac{1}{\cancel{X_{t+1}}\tilde{C}_{t+1}}\left(\alpha \frac{\tilde{Y}_{t+1}\cancel{X_{t+1}}}{\tilde{K}_t \cancel{X_{t+1}}} + 1 - \delta\right)\right]$$

$$\Rightarrow \frac{\gamma}{\tilde{C}_t} = \beta E_t\left[\frac{1}{\tilde{C}_{t+1}}\left(\alpha \frac{\tilde{Y}_{t+1}}{\tilde{K}_t} + 1 - \delta\right)\right]$$

② $$H_t^{\eta} = \frac{1}{\cancel{X_t}\tilde{C}_t}(1-\alpha)\frac{\cancel{X_t}\tilde{Y}_t}{H_t}$$

$$\Rightarrow H_t^{\eta} = \frac{1}{\tilde{C}_t}(1-\alpha)\frac{\tilde{Y}_t}{H_t}$$

③ $$\cancel{X_t}\tilde{Y}_t = Z_t(\cancel{X_t}\tilde{K}_{t-1})^{\alpha}(\cancel{X_t}H_t)^{1-\alpha}$$

$$\Rightarrow \tilde{Y}_t = Z_t \tilde{K}_{t-1}^{\alpha} H_t^{1-\alpha}$$

④ $$\gamma \frac{X_{t+1}}{\cancel{X_t}}\tilde{K}_t = (1-\delta)\frac{\cancel{X_t}}{\cancel{X_t}}\tilde{K}_{t-1} + \frac{\cancel{X_t}}{\cancel{X_t}}\tilde{I}_t$$

$$\Rightarrow \gamma \tilde{K}_t = (1-\delta)\tilde{K}_{t-1} + \tilde{I}_t$$

⑤ $$\cancel{X_t}\tilde{C}_t + \cancel{X_t}\tilde{I}_t = \cancel{X_t}\tilde{Y}_t$$

Q. why X_t should be inside the bracket w/ H_t ?

$$Y_t = Z_t X_t K_{t-1}^{\alpha} H_t^{1-\alpha}$$

$$w_t = (1-\alpha)\frac{Y_t}{H_t}$$ grows at rate $\gamma^{1+\alpha}$

grows at rate γ

$$= (1-\alpha)\frac{Z_t X_t K_{t-1}^{\alpha} H_t^{1-\alpha}}{H_t}$$

not aligns.

(4) Jan. 19 (Thu)

Special case $\underline{\underline{\delta = 1.}}$

$$\begin{cases} \dfrac{1}{C_t} = \beta E_t\left[\dfrac{R_{t+1}}{C_{t+1}}\right] \\ \varphi H_t^{\eta} = \dfrac{W_t}{C_t} \\ Y_t = Z_t K_{t-1}^{\alpha} H_t^{1-\alpha} \\ W_t = (1-\alpha)\dfrac{Y_t}{H_t} \\ R_t = \alpha \dfrac{Y_t}{K_{t-1}} \end{cases}$$

Guess & Verify.

$K_t = I_t = s Y_t$ $\qquad$ s: unknown (potentially fn. of parameters $\alpha, \beta, \cdots$)

$$C_t = (1-s) Y_t$$

$$\varphi H_t^{\eta} = \frac{(1-\alpha) Y_t / H_t}{(1-s) Y_t}$$

Set $\eta = 0$.

$$\varphi = \frac{(1-\alpha)}{(1-s) H_t} \implies H_t = \frac{1-\alpha}{\varphi(1-s)}$$

$\rightarrow$ Constant (i.e. ppl provide a fixed level of labor)

($\because$ income & static substitution effects cancel out)

$\uparrow$ due to $\delta = 1$.

$$\frac{1}{\cancel{(1-s) Y_t}} = \beta E_t\left[\frac{\alpha Y_{t+1} / \underbrace{K_t}_{= sY_t}}{(1-s) Y_{t+1}}\right]$$

$$= \beta E_t\left[\frac{\alpha}{s \cancel{(1-s) Y_t}}\right]$$

$$\implies \underline{s = \alpha\beta.}$$ intuitive!

Value fn. Iteration

Deterministic growth model

$$\underset{n\times 1}{K} = \begin{array}{c} \begin{array}{cccc} K_1 & K_2 & \cdots & K_n \end{array} \\ \{5.1,\ 5.2,\ \cdots,\ 10.5\}' \end{array}$$

→ has to include the steady state

$$\underset{n\times 1}{V^0} = \begin{bmatrix} V^0(K_1) \\ \vdots \\ V^0(K_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

1st iteration

$$\underset{n\times 1}{V^1(K)} = \begin{bmatrix} \tilde{V}^1(K_1) \\ \tilde{V}^1(K_2) \\ \vdots \\ \tilde{V}^1(K_n) \end{bmatrix}$$

For each K_i, $i=1,\cdots,n$

$$V^1(K_i) = \max_{K'} \left[\ln(K_i^\alpha - K' + (1-\delta)K) + \beta \underbrace{V^0(K')}_{=0} \right]$$

since $\ln(\cdot)$ is monotonic. — optimal $K_1, K_2, \cdots, \text{or } K_n$

min. possible investment — max. possible investment

(we impose the max. to be on one of the points in K)

→ basically finding the K' among $K_1, \cdots, K_n$ that maximizes $V^1(K_i)$

$\Rightarrow \tilde{V}^1(K_i)$ (the maximized $V^1(\cdot)$)

$$\underset{n\times 1}{V^2(K)} = \begin{bmatrix} \tilde{V}^2(K_1) \\ \vdots \\ \tilde{V}^2(K_n) \end{bmatrix}$$

$\forall i=1,\cdots,n$, find K' that solves

$$V^2(K_i) = \max_{K'} \left[\ln(K_i^\alpha - K' + (1-\delta)K_i) + \beta V^1(K') \right]$$

find the optimal K'

Then, compare $V^1(K)$ and $V^2(K)$.

↳ How?

$$\max \left\| \begin{bmatrix} \tilde{V}^1(K_1) - \tilde{V}^2(K_1) \\ \vdots \\ \tilde{V}^1(K_n) - \tilde{V}^2(K_n) \end{bmatrix} \right\| \quad \text{(if)} < \varepsilon = 0.01$$

then $\tilde{V}^1$ & $\tilde{V}^2$ close enough, hence we found solution!

Stochastic growth model

$V^1(K,z) \quad \forall\, K_\ell, z_m \quad \ell = 1, \cdots, n, \; m = 1, \cdots, n_z$

$$V^1(K_\ell, z_m)$$
$$= \max_{K'} \left[\ln\left(z_m K_\ell^\alpha - K' + (1-\delta)K_\ell\right) + \beta \sum_{j=1}^{n_z} \pi_{m,j} V^0(K', z_j) \right]$$

Deterministic Growth Model

$$V(k) = \max_{K', H} \left[\ln(K^\alpha H^{1-\alpha} - K' + (1-\delta)K) - \varphi \frac{H^{1+\eta}}{1+\eta} + \beta V(K') \right]$$

① Choose n number of capital grids

$$K = \{K_1, K_2, \cdots, K_n\}$$

② Make initial guess $V^0(K) = \begin{bmatrix} \\ \\ \end{bmatrix}$

③ Choose K' that maximizes $V'(K)$

$$V'(K) = \max_{K', H} \left[\ln(K^\alpha H^{1-\alpha} - K' + (1-\delta)K - \varphi \frac{H^{1+\eta}}{1+\eta} + \beta V^0(K) \right]$$

for each K_i, look for $\tilde{K}'$ that maximizes $V'(K_i)$

Compare

$$V'(K_i) = \left[\ln(K_i^\alpha (H^*)^{1-\alpha} - K_i + (1-\delta)K_i) - \varphi \frac{(H^*)^{1+\eta}}{1+\eta} + \beta V^0(K_1) \right]$$

where H^* is a sol. to

$$\frac{1}{c}(1-\alpha)\left(\frac{K_i}{H^*}\right)^\alpha = \varphi (H^*)^\eta$$

Do this for other possible $K' = K_2, \cdots, K_n$.

(p.15) Solving for the steady state.

$$\frac{1}{\bar{C}} = \frac{1}{\bar{C}}\beta(\bar{R}+1-\delta) \rightarrow \boxed{\bar{R} = \frac{1}{\beta}+\delta-1}$$

$$\bar{R} = \alpha\frac{\bar{Y}}{\bar{K}} = \alpha\frac{\bar{K}^{\alpha}\bar{H}^{1-\alpha}}{\bar{K}} = \alpha\left(\frac{\bar{K}}{\bar{H}}\right)^{\alpha-1} \rightarrow \boxed{\frac{\bar{K}}{\bar{H}} = \left(\frac{\frac{1}{\beta}+\delta-1}{\alpha}\right)^{\frac{1}{\alpha-1}} \equiv \Omega}$$

$$\bar{W} = 1-\alpha\frac{\bar{Y}}{\bar{H}} = (1-\alpha)\frac{\bar{K}^{\alpha}\bar{H}^{1-\alpha}}{\bar{H}} = (1-\alpha)\left(\frac{\bar{K}}{\bar{H}}\right)^{\alpha} = (1-\alpha)\Omega^{\alpha}$$

$$\frac{1}{\bar{C}}\bar{W} = \varphi\bar{H}^{\eta}$$

$$\frac{1}{\bar{C}}(1-\alpha)\Omega^{\alpha} = \varphi\bar{H}^{\eta} \rightarrow \boxed{\bar{C} = \frac{(1-\alpha)\Omega^{\alpha}}{\varphi\bar{H}^{\eta}}}$$

$$\Rightarrow \bar{H} = \left[\frac{1-\alpha}{\varphi(1-\delta\Omega^{1-\alpha})}\right]^{\frac{1}{1+\eta}}$$

(p.16) Log-linearization.

$$\hat{X}_t = \ln X_t - \ln\bar{X} \Rightarrow X_t = \bar{X}e^{\hat{X}_t} \approx \bar{X}(1+\hat{X}_t)$$

$$\left.\begin{aligned} X_t &= \bar{X}\cdot\left(\frac{X_t}{\bar{X}}\right) \\ &= \bar{X}\cdot e^{\ln\left(\frac{X_t}{\bar{X}}\right)} \\ &= \bar{X}\cdot e^{\hat{X}_t} \end{aligned}\right\} \Rightarrow \text{1st order Taylor approx.}$$

Recall $f(x) \approx f(a) + f'(a)(x-a)$

$$e^{x} \approx e^{0} + e^{0}(x-0) = 1+x$$

$$\boxed{X_t \approx \bar{X}(1+\hat{X}_t)}$$

$$Y_t = Z_t K_{t-1}^{\alpha} H_t^{1-\alpha}$$

Note:
$$X_t^a = \bar{X}^a e^{a\hat{x}_t}$$
$$\approx \bar{X}^a (1 + a\hat{X}_t)$$

$$\cancel{\bar{Y}}(1+\hat{Y}_t) = \bar{Z}(1+\hat{Z}_t)\bar{K}^{\alpha}(1+\alpha\hat{K}_{t-1})\bar{H}^{1-\alpha}(1+(1-\alpha)\hat{H}_t)$$

$$= \bar{Z}\bar{K}^{\alpha}\bar{H}^{1-\alpha}(1+\hat{Z}_t)(1+\alpha\hat{K}_{t-1})(1+(1-\alpha)\hat{H}_t)$$

$$= \underbrace{\bar{Z}\bar{K}^{\alpha}\bar{H}^{1-\alpha}}_{=\cancel{\bar{Y}}}(1+\hat{Z}_t+\alpha\hat{K}_{t-1}+(1-\alpha)\hat{H}_t)$$

Note: $(1+\hat{X}_t)(1+\hat{Y}_t)$
$$= 1 + \hat{X}_t + \hat{Y}_t + \underbrace{\hat{X}_t\hat{Y}_t}_{\approx 0}$$
$$\approx 1 + \hat{X}_t + \hat{Y}_t$$

$$\Rightarrow \boxed{\hat{Y}_t = \hat{Z}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{H}_t}.$$

<Log-linearizing the EC>

$$\frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}}(R_{t+1}+1-\delta)\right]$$

$$LHS = \cancel{C^{-1}}(1-\hat{C}_t)$$

$$RHS = \beta E_t\left[C^{-1}(1-\hat{C}_{t+1})\{\bar{R}(1+\hat{R}_{t+1})+1-\delta\}\right]$$

$$= \beta E_t\left[C^{-1}\bar{R}(1-\hat{C}_{t+1})(1+\hat{R}_{t+1}) + C^{-1}(1+\hat{C}_{t+1})(1-\delta)\right]$$

$$= \beta E_t\left[\cancel{C^{-1}}\bar{R}(1-\hat{C}_{t+1}+\hat{R}_{t+1}) + \cancel{C^{-1}}(1-\delta)(1+\hat{C}_{t+1})\right]$$

$$\Rightarrow \cancel{1} - \hat{C}_t = \beta E_t\left[\bar{R}(1-\hat{C}_{t+1}+\hat{R}_{t+1}) + (1-\delta)(1+\hat{C}_{t+1})\right]$$

$$= \cancel{\beta} E_t\left[\underbrace{\bar{R}+1-\delta}_{=\frac{1}{\beta}} - \underbrace{(\bar{R}+1-\delta)}_{=\frac{1}{\cancel{\beta}}}\hat{C}_{t+1} + \bar{R}\hat{R}_{t+1}\right]$$

$$= E_t\left[\cancel{1} - \hat{C}_{t+1} + \beta\bar{R}\hat{R}_{t+1}\right]$$

$$\Rightarrow \boxed{\hat{C}_t = E_t\left[\hat{C}_{t+1} - \beta\bar{R}\hat{R}_{t+1}\right]}$$

$$\varphi H_t^{\eta} = \frac{w_t}{C_t} \implies \varphi \bar{H}^{\eta}(1+\eta \hat{H}_t) = \bar{w}(1+\hat{w}_t)\,\bar{C}^{-1}(1-\hat{C}_t)$$

(6) Jan. 26 (Thu)

$$u = \ln C_t - \varphi \frac{h_t^{1+\eta}}{1+\eta}$$

SP problem (#4-4)

$$\mathcal{L} = \underline{\hspace{4cm}} + \lambda_t \left[C_t - \pi_t C_{1,t} - (1-\pi_t) C_{0,t} \right]$$

FONC wrt $C_{1,t}$

$$\pi_t U'(C_{1t}, 1-\hat{h}) - \lambda_t \pi_t = 0 \Rightarrow U'(C_{1t}, 1-\hat{h}) = \lambda_t.$$

FONC wrt $C_{0,t}$

$$(1-\pi_t) U'(C_{0t}, 1) - \lambda_t (1-\pi_t) = 0 \Rightarrow U'(C_{0t}, 1) = \lambda_t$$

$$\Rightarrow U'(C_{1t}, 1-\hat{h}) = U'(C_{0t}, 1) = \lambda_t$$

If $U(C, \ell) = V(C) + W(\ell)$ (e.g. $U(c, \ell) = \ln c - \varphi \frac{(1-\ell)^{1+\eta}}{1+\eta}$

$$\Rightarrow V'(C_{1t}) = V'(C_{0t})$$

$$\Rightarrow \underline{C_{1t} = C_{0t} \text{ at optimum}}$$

Ex ante Eu.

$$EU(C_t, l_t) = \pi_t[u(C_t) + v(1-\hat{h})] + (1-\pi_t)[u(C_t) + v(1)]$$

(Assume $v(l) = A\ln(l)$)

$$= u(C_t) + \pi_t A \ln(1-\hat{h})$$

(Assume large number of agents, then per capita hours worked is $H_t = \pi_t \hat{h}$

$$= u(C_t) + \cancel{\pi_t} A \ln(1-\hat{h}) \frac{1}{\cancel{\pi_t}\hat{h}} \cdot H_t$$

$$= u(C_t) + A\frac{\ln(1-\hat{h})}{\hat{h}} H_t$$

$$= u(C_t) - \underbrace{\left[-A\frac{\ln(1-\hat{h})}{\hat{h}}\right]}_{B>0} H_t$$

$$= \boxed{u(C_t) - BH_t} \quad \cdots \; ☆$$

The ex ante preference looks like standard pref. w/ inverse Frisch elasticity $\eta = 0$. $(\eta^{-1} = \infty)$

Large family w/ continuum of family members.
Household head maximizes utility of members.
This implies that the rep. household's utility can be written as ☆

5. RBC w/ fiscal shocks

P.4 RC

$$c_t + i_t \leq r_t k_{t-1} + w_t h_t + T_t (r_t - \delta) K_{t-1} + \varphi_t w_t H_t - g_t$$
$$- T_t (r_t - \delta) k_{t-1} - \varphi_t w_t h_t$$

Impose mkt clearing: $c_t = C_t$

$i_t = I_t$

$k_{t-1} = K_{t-1}$

$$C_t + I_t = r_t K_{t-1} + w_t H_t + \cancel{T (r_t - \delta) K_{t-1}} + \cancel{\varphi_t w_t H_t}$$
$$- g_t - \cancel{T (r_t - \delta) K_{t-1}} - \cancel{\varphi_t w_t H_t}$$

using $r_t K_{t-1} + w_t H_t = Y_t$.

$$C_t + I_t = Y_t - g_t \Rightarrow \underline{Y_t = C_t + I_t + g_t}$$

(8) Feb. 2 (Thu)

Ch. 5

Slide #7

- Recursive formulation

  H's problem :

$$V(k,K,v) = \max_{k',c,h} \left[u(c+\pi g, h) + \beta E V(k',K',v') \right]$$

s.t. $c + i \leq rk + wh + \xi - \tau(r-\delta)k - \varphi wh$

Recursive C.E.

(i) $V(k,K,v)$
$c(\;\;"\;\;)$
$h(\;\;"\;\;)$
$i(\;\;"\;\;)$

(ii) $H^d(K,v)$
$K^d(K,v)$

(iii) $w(K,v)$
$r(K,v)$

(iv) $\xi(K,v)$

such that

- Household optimization
- Firm "
- Gov't BC satisfied
- Market clearing:
  - $h = H = H^d$
  - $k = K = K^d$
  - $c + i + g = zK^\alpha H^{1-\alpha}$
- Consistency

  $m(K,v) = (1-\delta)K + i(K,k,v)$

Direct Attack

Sequential formulation

(H)'s problem

$$\mathcal{L} = E \sum_{t=0}^{\infty} \beta^t \Big[\{ u(c_t + \pi g_t, h_t) \} .$$

$$+ \lambda_t \{ r_t k_{t-1} + w_t h_t + \xi_t - \tau_t (r_t - \delta) k_{t-1} - \varphi_{t-1} w_t h_t - c_t - i_t \} \Big]$$

$k_t + (1-\delta) k_{t-1}$

(motion of capital)

FONCs

$[c_t]$ $\lambda_t = u_1(c_t + \pi g_t, h_t)$

$[h_t]$ $\lambda_t (w_t - \varphi_t w_t) = u_2(c_t + \pi g_t, h_t)$

$$\Rightarrow \lambda (1 - \varphi_t) w_t = u_2(c_t + \pi g_t, h_t)$$

difference from before. (RBC)

$[k_t]$ $\lambda_t = \beta E_t [\lambda_{t+1} \{ r_{t+1} - \tau_{t+1} (r_{t+1} - \delta) + 1 - \delta \}]$

$$\Rightarrow \lambda_t = \beta E_t [(1 - \tau_{t+1})(r_{t+1} - \delta) + 1 \}]$$

diff. from RBC

(F)'s prob.

$$\begin{cases} r_t = \alpha \dfrac{Y_t}{K_t^d} \\ w_t = (1-\alpha) \dfrac{Y_t}{H_t^d} \end{cases}$$

Impose mkt clearing

$h_t = H_t^d = H_t$

$k_{t-1} = K_t^d = K_{t-1}$

$C_t + I_t + g_t = z_t K_{t-1}^{\alpha} H_t^{1-\alpha}$

Eqm. Conditions

- $u_1(c_t + \pi g_t, H_t)(1 - \varphi_t) w_t = u_2(c_t + \pi g_t, H_t)$

$$u_1(c_t + \pi g_t, H_t) = \beta E_t [u_1(c_{t+1} + \pi g_{t+1}, H_{t+1})(1 - \tau_{t+1})(r_{t+1} - \delta) + 1]$$

$$w_t = (1-\alpha) \frac{Y_t}{H_t}$$

$$r_t = \alpha \frac{Y_t}{K_{t-1}}$$

$$+ \begin{cases} c_t + I_t + g_t = Y_t \\ K_t = (1-\delta) K_{t-1} + I_t \\ Y_t = z_t K_{t-1}^{\alpha} H_t^{1-\alpha} \end{cases}$$

Recursive formulation

Ⓗ's prob.

$$V(k,K,V) = \max_{k',c,h} \left[u(c+\pi g, h) + \beta E V(k',K',V') \right]$$

$$\text{s.t.} \quad c + i = rk + wh + \xi - \tau(r-\delta)k - \varphi wh$$

$$k' = (1-\delta)k + i$$

$$= \max_{k',h} \left[u(rk + wh + \xi - \tau(r-\delta)k - \varphi wh - k' + (1-\delta)k + \pi g, h) + \beta E V(k',K',V') \right]$$

By Envelope condition,

$$V_1(k,K,V) = u_1(\ )(r - \tau(r-\delta) + 1 - \delta)$$

FONCs

$$[k'] \quad u_1(c+\pi g, h) = \beta E V(k',K',V')$$

$$\Rightarrow \boxed{u_1(c+\pi g, h) = \beta E\left[u_1(c'+\pi g', h') \times \{(1-\tau')(r'-\delta)+1\} \right]}$$

$$[h] \quad u_1(c+\pi g, h)(w - \varphi w) + u_2(c+\pi g, h) = 0$$

$$\Rightarrow \boxed{u_1(c+\pi g, h)(1-\varphi)w + u_2(c+\pi g, h) = 0}$$

Ch. 6

CES utility fn.

$$C = \left[C_1^{\frac{1-e}{e}} + C_2^{\frac{1-e}{e}} \right]^{\frac{e}{1-e}}$$

Optimization:

$$\max_{C_1, C_2} C = \left[C_1^{\frac{1-e}{e}} + C_2^{\frac{1-e}{e}} \right]^{\frac{e}{1-e}} \quad \text{s.t. } P_1C_1 + P_2C_2 = E$$

FONCs

$[C_i]$ $\quad \frac{e}{e-1}\left[C_1^{\frac{e-1}{e}} + C_2^{\frac{e-1}{e}} \right]^{\frac{1}{e-1}} \frac{e-1}{e} C_i^{\frac{-1}{e}} = \lambda P_i \qquad i = 1, 2$

$$\Rightarrow \left[\frac{C_2}{C_1}\right]^{\frac{1}{e}} = \frac{P_1}{P_2}$$

$$\Rightarrow \frac{C_2}{C_1} = \left[\frac{P_1}{P_2}\right]^{-e}$$

elasticity of substitution : e

high e : goods are more substitutable.

low e : goods are more complementary