204C 2nd Midterm Preparation

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1. **[2008,2016] The revelation principle:**

   A. In ordinary language, state the revelation principle.

   If there exists a direct mechanism $g(m)$ that implements the social choice function $f(\theta)$, then there exists a direct mechanism that is truthfully implementable. Thus any non-truthful direct mechanism can be replicated by a truthful direct mechanism. That is, if the designer can implement $f(\theta)$ when player i lies, the designer can have i tell the truth and lie for i.

   B. Again in ordinary language, explain why it is important (explain both the necessary and sufficient implications).

   - There may be many indirect mechanisms that are implementable, but we only need to look at direct mechanism that are truthfully implementable.
   - On the other side, if we cannot find a direct mechanism that is truthfully implementable, then there is no way to implement the social choice function. That is, there is not an indirect and/or non-truthfully implementable mechanism either.

2. **[2008]** Suppose that there are only two types θ_1 and θ_2, and that column and row draw these independently. $\theta_i|\theta_j$ means that a person who is of type j announces that she is of type i. Payoff is row, column.

   A. Provide a matrix example where truth is a dominant strategy equilibrium.

	$\theta_1\|\theta_1$	$\theta_2\|\theta_1$	$\theta_1\|\theta_2$	$\theta_2\|\theta_2$
$\theta_1\|\theta_1$	2,2	3,1	10,12	3,14
$\theta_2\|\theta_1$	1,3	2,2	9,6	1,7
$\theta_1\|\theta_2$	4,5	6,4	11,15	4,16
$\theta_2\|\theta_2$	5,2	7,1	12,14	5,16

   B. Suppose that the probability of a draw for each type is 1/2. Provide a matrix example where truth is a Bayesian-Nash equilibrium, but not dominant strategy equilibrium.

	$\theta_1\|\theta_1$	$\theta_2\|\theta_1$	$\theta_1\|\theta_2$	$\theta_2\|\theta_2$
$\theta_1\|\theta_1$	0,0	3,1	8,15	10,14
$\theta_2\|\theta_1$	1,1	4,2	9,8	5,7
$\theta_1\|\theta_2$	4,5	8,6	14,18	4,14
$\theta_2\|\theta_2$	3,6	7,1	12,10	16,16

An example where truth telling is a Nash Equilibrium but not a dominant-strategy equilibrium.

	$\theta_1\|\theta_1$	$\theta_2\|\theta_1$	$\theta_1\|\theta_2$	$\theta_2\|\theta_2$
$\theta_1\|\theta_1$	2,2	3,1	8,12	3,14
$\theta_2\|\theta_1$	1,1	4,2	9,8	1,7
$\theta_1\|\theta_2$	4,5	8,6	14,18	4,14
$\theta_2\|\theta_2$	5,2	7,1	12,14	5,16

3. **[2016] Bridge Problem:** Suppose the the cost of building a bridge is \$90K and if the bridge is built, Player 1 will have a tax payment (not transfer payment) of \$60K, Player 2 will have a tax payment of \$30K and Player 3 will pay nothing. Suppose that the true value to Players 1, 2, and 3 is \$40K, \$30K, and \$40K, respectively. Under the VCG mechanism,

   A. Will the bridge be built?

   Yes, since the total benefit of building the bridge to all three players (\$110K) is greater than the cost (\$90K).

   B. What will be the transfer payments? Show your calculations.

   Assuming everyone is telling the truth, the VCG mechanism requires the transfer payment of j to be

$$t^j(X,N,M) = \sum_{i\neq j}^{N} v^i(\hat{x}(X,N-1,M^{-j}),m_i) - \sum_{i\neq j}^{N} v^i(\hat{x}(X,N,M),m_i).$$

   Then the transfer payments can be calculated as

$$\begin{aligned} t^1 &= [0+0] - [(30-30)+(40-0)] = \quad -\$40K \\ t^2 &= [0+0] - [(40-60)+(40-0)] = \quad -\$20K \\ t^3 &= [0+0] - [(40-60)+(30-30)] = \quad +\$20K \end{aligned}$$

   That is, players 1 and 2 will receive \$40K and \$20K,respectively, from the government, and player 3 should pay the government \$20K.

   C. Why is truth a dominant-strategy equilibrium? You can explain in words.

   - The buyer's report of his value does not determine his transfer payments (or tax payments) but only whether the bridge is built. If the buyer overreports his value, he may end up buying the item for more than it is worth (i.e. true benefit < (reported) cost to others < reported benefit). If the buyer underreports his value, he may lose out from gaining a surplus (i.e. reported benefit < reported cost to others < true benefit). All other situations yield the same surplus to the buyer whether or not the buyer tells the truth. So telling the truth is a weakly dominant strategy, and this is true for all players.
   - Truth is a dominant-strategy equilibrium because the marginal costs and benefits to society are identical to the costs and benefits to player j. Player j wants the bridge to be built if and only if his <u>true</u> benefit is greater than the reported cost to the other players. The bridge will be built if and only if player j's <u>reported</u> benefit is greater than the reported cost to the other players.

4. **[2016] VCG Mechanism:** Briefly point out two problems with the VCG mechanism.

   (1) It need not result in a balanced budget; equivalently, utility may be lost in eliciting the truth.

   (2) The rule can be gamed by coalitions (i.e. collusions); players can exaggerate their reported values to increase the size of their transfer payments.

   (3) The players might choose not to participate

5. Check again! **[2009]** For $v^i(f(\theta), \theta)$ quasi-linear, the social choice function $f(\theta)$ is ex-post efficient if: α. $\sum v^i(f(\theta), \theta) \geq \sum v^i(x, \theta)$ for all x, and β. $\sum t^i(\theta) = 0$. That is, there is balanced budget. Show that α need not hold if $v^i(f(\theta), \theta)$ is not quasi-linear.
   Consider a two-person world with only one good—apples. Suppose we have a non-quasi-linear utlity function where person 1's utility is $u^1 = 1A$, and person 2's utility is $u^2 = 2A$ where A is the number of apples. Let x be to distribute 10 apples to person 1 and 0 apples to person 2. And let $f(\theta)$ be to distribute 0 apples to person 1 and 9 apples to person 2. Even though the sum of utilities is greater with $f(\theta)$ than with x, there is no way to redistribute the apples (let's call it x') so that both players 1 and 2 weakly prefer x' to x (that is, there is no x' Pareto superior to x).

6. **[2008]** Complete the following definition.

   For $v^i(f(\theta), \theta)$ quasi-linear, the social choice function $f(\theta)$ is ex-post efficient if:
   α. $\sum v^i(f(\theta), \theta) \geq \sum v^i(x, \theta)$ for all x, and
   β. $\sum t^i(\theta) = 0$ (balanced budget)

7. **[2008] Second Price auction** (assume that the seller does not value the item).

   A. Define a second-price auction.
   Each player calls out his/her type m_i or bids B_i. The player with the highest value/bid receives the item but is only required to pay the seller the second-highest declared bid/value.

   B. Prove that it is truthfully implementable in dominant strategies. Ordinary language is acceptable, but do not "prove" by example (a 2-player example is not adequate, but an n-player example is).
   In this auction, it never hurts to tell the truth, but it may hurt to tell a lie. Let's consider the cases where telling a lie would lead to change in outcomes.
   - The only case that player j's strategy of overreporting can change the outcome is when another player has a value higher than j's true value but less than j's reported value. In such case, player j would win the item but has to pay more than his true value. (j's true value $<$ another player's value $<$ j's reported value)
   - Underreporting would also be an inappropriate strategy. The outcome would be different only when another player reported more than player j's message but less than j's true value. In such case, player j would lose out on a positive gain. (j's reported value $<$ another's value $<$ j's true value)

8. **[2009,2016]** Suppose that there are several buyers with quasi-linear utility functions. Is there a dominant strategy mechanism that extracts all the surplus from the buyer with the highest value? Yes or No. Prove your answer to be the case.

   - No. To be less abstract, we consider a first-price sealed-bid auction, which is institution free. By the revelation principle if we show that we cannot find a direct mechanism that is truthfully implementable, it is not implementable at all.
   - It can be easily seen that truth telling is not a dominant strategy equilibrium in this case. Suppose we have two buyers and buyer 2's bid is $\hat{B}$ such that $\hat{B} < \theta_1$. Then buyer 1 has incentive to untruthfully choose B_1 such that $\hat{B} < B_1 < \theta_1$, because he can obtain a surplus.
   - We can also easily demonstrate that truth telling is not a Bayesian Nash equilibrium. Suppose buyer 2 has told the truth (but 2's value is unknown to 1). Then buyer 1 will maximize $[\theta_1 - m_1]P(m_1 > \theta_2)$. Assuming θ_2 is uniformly distributed on [0,1], this is equivalent to $[\theta_1 - m_1]m_1$. The FOC is $m_1 = \frac{1}{2}\theta_1$.
   - In words, player 1 needs to balance the tradeoff between the probability of winning the item and the surplus. The optimal strategy in this case is to announce one-half of her true value. A symmetric argument holds for player 2.
   - In essence, the social choice function cannot be implemented because the seller tries to extract all of the surplus value from the winning buyer.

9. **[2008, 2009] Bilateral Trade.**

   Suppose that there is a buyer and a seller, and that each side values the item somewhere between 0 and 1. The values (utilities) that the buyer and seller have for the item are θ_b, θ_s, respectively. Both sides have quasi-linear utilities.

   A. What is the VCG mechanism in this case? Explain the mechanism both when there is a trade and when there is no trade.

      (The VCG mechanism can accomplish a mechanism where a trade takes place if and only if $\theta_b > \theta_s$.)
      - If $m_b \leq m_s$, there will be no trade and the utility of the seller will be θ_s and the utility of the buyer will be 0.
      - If $m_b > m_s$, there will be a trade and the transfer payments will be $t^b = m_s$ and $t^s = -m_b$. The utilities will be $u^b = \theta_b - m_s$ and $u^s = 0 - (-m_b) = m_b$.

   B. Show that truth-telling is a dominant strategy equilibrium.

      - The buyer wants to trade if and only if $\theta_b > m_s$, but a trade takes place if and only if $m_b > m_s$. So the buyer will always want to tell the truth (i.e. truth is a weakly dominant strategy).
      - The seller wants to trade if and only if $\theta_s < m_b$, but a trade takes place if and only if $m_s < m_b$. So the seller will always want to tell the truth (i.e. truth is a dominant strategy).
      - After all, one's announcement has no influence on the amount of own's transfer but only on whether a transfer will take place.

   C. Show that the mechanism is efficient.

      All Pareto improving trades will take place under this mechanism. (When it comes to bilateral trade, only the VCG mechanism is α efficient.)

   D. Point out one problem with this mechanism.

      The VCG mechanism does not satisfy balanced budget and one or both players may refuse to participate.

10. **[2009,2016] Gibbard-Satterthwaite:** If $f(\theta)$ has universal domain and the range has at least three elements, then the social choice function is truthfully implementable in dominant strategies if and only if it is dictatorial. Provide an example illustrating both aspects of this proposition (if and only if) along with an accompanying explanation. Note that quasi-linearity is not being assumed.

    Suppose we have two persons ($i = 1, 2$) and three alternatives of choice ($X = \{x, y, z\}$). Person 1 has only one type ($\Theta_1 = \{\theta_1\}$) and Person 2 has two types ($\Theta_2 = \{\theta_2', \theta_2''\}$). The rank ordering of x, y, and z is as follows

	θ_1	θ_2'	θ_2''
1	x	z	y
2	y	y	x
3	z	x	z

    Consider the ex post efficient social function $f(\cdot)$ with $f(\theta_1, \theta_2') = y$ and $f(\theta_1, \theta_2'') = x$.
    If person 2 is type θ_2'', she has incentive to claim to be θ_2' in order to obtain her most desired outcome y. So this social choice function is not truthfully implementable.
    Suppose instead that only person 2's preference counted (i.e. the social choice function is dictatorial). Then person 2 will declare here true type since she will get what she prefers the most. Since person 1 would have no effect, he would be indifferent between telling the truth and lying. Hence, telling the truth would be a weakly dominant strategy. Similarly, if person 1 were the dictator, then person 1 would choose his most preferred position and there would be no benefit to person 1 or person 2 from lying.

11. **[2009]** Let $V^* = V^i(\theta_i, \theta_i) = u^i(f(\theta_i, \theta_{-i}), \theta_i)$. Suppose $\theta_i \in [0, 1]$.
    State the Hotelling's Lemma.
    If $V^* = V^i(\theta_i, \theta_i) = u^i(f(\theta_i, \theta_{-i}), \theta_i)$ is absolutely continuous, then

    (a) $V^*(\theta_i) = V^i(\theta_i, \theta_i) = u^i(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) = u^i(f(0, \theta_{-i}), 0, \theta_{-i}) + \int_0^{\theta_i} u_2^i(f(s, \theta_{-i}), s, \theta_{-i})ds$

    (b) If V^* is differentiable at θ_i, then $V^*(\theta_i) = V_2^i = u_2^i$.

    Recall that $V(message,\ true\_value)$. If truth telling (i.e. $m_i = \theta_i$) is a Bayesian Nash equilibrium, then $\arg\max_{m_i} V^i(m_i, \theta_i) = \theta_i$; so, $V_1^i(\theta_i, \theta_i) = 0$

12. **[2008,2016] The revenue equivalence theorem.**

HOTELLING'S LEMMA: Suppose that $\theta_i \in [0,1]$. If $V^*(\theta_i) = V^i(\theta_i, \theta_i) = u^i(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})$ is absolutely continuous, then

(a) $V^*(\theta_i) = V^i(\theta_i, \theta_i) = u^i(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) = u^i(f(0, \theta_{-i}), 0, \theta_{-i}) + \int_0^{\theta_i} u_2^i(f(s, \theta_{-i})s, \theta_{-i})ds$

(b) If V^* is differentiable at θ_i, then $V^{*'}(\theta_i) = V_2^i = u_2^i$

Assume the following utility function: $u^i(f(m_i, \theta_{-i}), \theta_i) = P^i(m_i, \theta_{-i})\theta_i - \bar{t}^i(m_i, \theta_{-i})$.

EXPECTED REVENUE EQUIVALENCE THEOREM. Let there be N risk neutral buyers. Each buyer's valuation, θ_i, is drawn independently from an interval [0,1] with strictly positive density, $g(\theta_i)$. Suppose that a given pair of Bayesian Nash equilibria of two different auction procedures are such that for every buyer i: (1) for each possible realization of θ_i buyer i has an identical probability of getting the good in the two auctions; and (2) buyer i has the same expected utility level in the two auctions when his valuation for the object is the lowest possible level. Then the equilibria of the two auctions generate the same expected revenue for the seller.
Prove the Theorem.
Assuming that all others tell the truth, buyer i's expected payoff is

$$V^i(m_i, \theta_i) = U^i(f(m_i, \theta_{-i}), \theta_i) = P^i(m_i, \theta_{-i})\theta_i - \bar{t}^i(m_i, \theta_{-i}),$$

which is absolutely continuous at the optimum (i.e. $m_i = \theta_i$) by assumption. Therefore we can apply the Hotelling's Lemma.
We will take the total derivative of i's expected payoff $V^i(m_i, \theta_i)$ with respect to θ_i and evaluate it at $m_i = \theta_i$. By part (b) of the Lemma, we know such a derivative exists.

$$dV^* = V_1^i dm_i + V_2^i d\theta_i = U_1^i f_1 dm_i + U_2^i d\theta_i = U_2^i d\theta_i,$$

where the last inequality holds since $V_1^i = U_1^i f_1 dm_i = 0$ at optimum. From this, we obtain

$$V^{*'} = \frac{dV^*}{d\theta_i} = U_2^i = P^i(m_i, \theta_{-i}).$$

Now, we can use part (a) to derive $V^*(\theta_i)$ and impose the result obtained above.

$$\begin{aligned} V^*(\theta_i) = V^i(\theta_i, \theta_i) = U^i(f(\theta_i, \theta_{-i}), \theta_i) &= U^i(f(0, \theta_{-i}), 0) + \int_0^{\theta_i} U_2^i(f(s, \theta_{-i}), s)ds \\ &= U^i(f(0, \theta_{-i}), 0) + \int_0^{\theta_i} P^i(s, \theta_{-i})ds. \end{aligned}$$

The first term is identical for any auction type (the expected utility of the lowest type is the same regardless of the auction type). Note the second term is totally independent of the payment mechanism (i.e. $\bar{t}^i(m_i, \theta_{-i})$). That is, buyer i's expected utility is independent of the (implementable) transfer function.
All auctions that satisfy the conditions of the theorem produce the same expected cost to the buyers and therefore the same expected revenue to the seller (which is the sum of the bidders' payments). Therefore, all auction types yield the same expected revenue.

13. **[2008] All-pay Sealed-bid Auction withe Private Values**

In this auction, the following bid function is a truthfully-implementable Bayesian-Nash equilibrium strategy (assuming independent draws from a uniform distribution on [0,1]):

$$b(\theta) = [(n-1)/n]\theta^n$$

From this information derive the seller's expected revenue.

The seller keeps all of the n bids. So the seller's expected revenue is

$$n\int_0^1 \frac{n-1}{n}\theta^n d\theta = (n-1)\frac{\theta^{n+1}}{n+1}\bigg|_0^1 = \frac{n-1}{n+1},$$

which is the same expected revenue as the first and second price auctions.

14. **[2009]** Let θ_i be independently and uniformly distributed on [0,1]. Consider a sealed-bid, private-value, all-pay auction. Let i's utility $= \theta_i - b_i$ if i gets the object and $-b_i$ if i does not get the object.

(a) Derive the (symmetric) Bayesian-Nash bidding strategy.

Since the player pays his bid whether he wins or not, the expected payoff of the player is

$$V(m,\theta) = \theta[G(m)]^{n-1} - b(m), \quad \text{where} \quad G(m) = \theta$$

$$V_1(m,\theta) = V_m(m,\theta) = (n-1)\theta[G(m)]^{n-2} - b'(m)\Big|_{m=\theta} = 0$$

$$\iff (n-1)\theta^{n-1} - b'(\theta) = 0$$

$$\iff b(\theta) = \frac{n-1}{n}\theta^n.$$

(b) From (a), derive the seller's expected revenue.

The seller keeps all of the n bids. So the seller's expected revenue is

$$n\int_0^1 \frac{n-1}{n}\theta^n d\theta = (n-1)\frac{\theta^{n+1}}{n+1}\bigg|_0^1 = \frac{n-1}{n+1},$$

which is the same expected revenue as the first and second price auctions.

(c) Show that the two necessary and sufficient conditions of the Bayesian incentive compatibility theorem are satisfied in this case.

(1) $V_1(\theta,\theta) + V_2(\theta,\theta)$ is increasing in θ. (monotonicity requirement)
Since $V_1(m,\theta) = 0$ at $m = \theta$, we only need to show that $V_2(m,\theta)$ is increasing in θ at $m = \theta$.

$$V_2(m,\theta) = V_\theta(m,\theta) = [G(m)]^{n-1} = \theta^{n-1}$$

$$\implies \frac{dV_2}{d\theta} = (n-1)\theta^{n-2} \geq 0.$$

Thus the monotonicity requirement is satisfied.

(2) $V^*(\theta) = V(\theta,\theta) = V(0,0) + \int_0^\theta V_2(m,s)ds = 0 + \int_0^\theta [G(s)]^{n-1}ds = \int_0^\theta s^{n-1}ds = \frac{1}{n}s^n\Big|_0^\theta = \frac{1}{n}\theta^n.$
This result is equivalent to that obtained in the first price auction.

15. **[2008] First-price Sealed-bid Common Values Auction**

Assume that $u^j = [\sum \theta_i]/n - b_j$ if J wins the auction and that θ_i is distributed independently and uniformly on [0,1]. Recall that the expected k^{th} highest value among n values independently drawn from the uniform distribution on [0,1] is $(n+1-k)/(n+1)$.

Derive the relevant equations for a truthful Bayesian Nash Equilibrium that one could then solve using Maple.

We can derive the BNE bidding strategies via our incentive compatibility approach. The expected value of the item is a weighted average of the bidder's own observation (θ) and the expected value of the $n-1$ bidders' observations ($\frac{m}{2}$).

$$V(m,\theta) = \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \mathbf{b}(m)\Big)[G(m)]^{n-1}, \quad G(m) = m \ (\because \text{ uniform distribution}).$$

$$V_m(m,\theta) = \Big(\frac{n-1}{2n} - \mathbf{b}'(m)\Big)m^{n-1} + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \mathbf{b}(m)\Big)(n-1)m^{n-2}\Big|_{m=\theta} = 0$$

$$\iff \Big(\frac{n-1}{2n} - \mathbf{b}'(\theta)\Big)\theta^{n-1} + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{\theta}{2} - \mathbf{b}(\theta)\Big)(n-1)\theta^{n-2} = 0$$

$$\iff \Big(\frac{n-1}{2n} - \mathbf{b}'(\theta)\Big)\theta + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{\theta}{2} - \mathbf{b}(\theta)\Big)(n-1) = 0$$

When $\theta = 0$,

$$\implies -\mathbf{b}(0)(n-1) = 0,$$

so $b(0) = 0$ as long as $n-1 > 0$.

16. **[2016]** Let θ_i be independently and uniformly distributed on [0,1]. Consider a sealed-bid, first-price common-values auction, where the common value is the average of the types. Derive the differential equations needed to solve the bidding function. Do not actually solve these equations expect for $b(0)$.

In order to avoid the winner's curse, bidder i will not bid her observation (as was the case in the private values auction) but bid her expected value conditional on everyone else having a weakly lower signal than her message.

We can derive the BNE bidding strategies via our incentive compatibility approach. The expected value of the item is a weighted average of the bidder's own observation (θ) and the expected value of the $n-1$ bidders' observations ($\frac{m}{2}$).

$$V(m,\theta) = \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \mathbf{b}(m)\Big)[G(m)]^{n-1}, \quad G(m) = m \ (\because \text{ uniform distribution}).$$

$$V_m(m,\theta) = \Big(\frac{n-1}{2n} - \mathbf{b}'(m)\Big)m^{n-1} + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \mathbf{b}(m)\Big)(n-1)m^{n-2}\Big|_{m=\theta} = 0$$

$$\iff \Big(\frac{n-1}{2n} - \mathbf{b}'(\theta)\Big)\theta^{n-1} + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{\theta}{2} - \mathbf{b}(\theta)\Big)(n-1)\theta^{n-2} = 0$$

$$\iff \Big(\frac{n-1}{2n} - \mathbf{b}'(\theta)\Big)\theta + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{\theta}{2} - \mathbf{b}(\theta)\Big)(n-1) = 0$$

When $\theta = 0$,

$$\implies -\mathbf{b}(0)(n-1) = 0,$$

so $b(0) = 0$ as long as $n-1 > 0$.

17. **[2009]** Let θ_i be independently and uniformly distributed on [0,1]. Consider a sealed-bid, second-price common-values auction (where the value of the object $= \sum \theta_i/N$). Derive the first order conditions for truth-telling. Note that for this question, you are not expected to do anything more than derive the first order conditions.

In order to avoid the winner's curse, bidder i will not bid her observation (as was the case in the private values auction) but bid her expected value conditional on everyone else having a weakly lower signal than her message.

We can derive the BNE bidding strategies via our incentive compatibility approach. The expected value of the item is a weighted average of the bidder's own observation (θ) and the expected value of the $n-1$ bidders' observations ($\frac{m}{2}$). The expected bid of the second highest bidder is the highest of the $n-1$ other players whose θ_i are uniformly distributed on $[0, m]$. (Recall that k-th highest observation in $\mathcal{U}[0,1]$: $\frac{n+1-k}{n+1}$).

$$
\begin{aligned}
V(m,\theta) &= \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \mathbf{b}(\frac{n-1}{n}m)\Big)[G(m)]^{n-1} \\
&= \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \mathbf{b}(\frac{n-1}{n}m)\Big)m^{n-1} \quad (\because \text{ uniform dist.}) \\
&= \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \frac{n-1}{n}\mathbf{b}(m)\Big)m^{n-1},
\end{aligned}
$$

where the last inequality holds by assuming a proportional relationship between the message and the bid.

Taking the FOC,

$$
V_m(m,\theta) = \Big(\frac{n-1}{2n} - \frac{n-1}{n}\mathbf{b}'(m)\Big)m^{n-1} + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{m}{2} - \frac{n-1}{n}\mathbf{b}(m)\Big)(n-1)m^{n-2}\Big|_{m=\theta} = 0
$$

$$
\begin{aligned}
\iff \Big(\frac{n-1}{2n} - \frac{n-1}{n}\mathbf{b}'(\theta)\Big)\theta^{n-1} + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{\theta}{2} - \frac{n-1}{n}\mathbf{b}(\theta)\Big)(n-1)\theta^{n-2} = 0 \\
\iff \Big(\frac{n-1}{2n} - \frac{n-1}{n}\mathbf{b}'(\theta)\Big)\theta + \Big(\frac{1}{n}\cdot\theta + \frac{n-1}{n}\cdot\frac{\theta}{2} - \frac{n-1}{n}\mathbf{b}(\theta)\Big)(n-1) = 0
\end{aligned}
$$

18. **[2008, 2016]** Consider an insurance company that sells accident insurance. The people who buy the insurance know their own type, $1-\theta$, which is the probability of having a loss L, where θ is an element in [0,1]. The insurance company would like to design a contract such that individuals truthfully reveal their types and that is actuarially fair. Let $S(\theta)$ be the share of the loss by the insurance company if there is an accident and $P(\theta)$ be the premium paid by the individual for the insurance (the premium is paid regardless of the state of the world). The insurance company provides and actuarially-fair policy for every m. Assume that utility of income has the following properties: $v(0) = 0$; $v' > 0$, and $v'' < 0$. Derive the first order conditions for incentive compatibility. Do not try to solve the differential equations.

The expected payoff is

$$V(m,\theta) = (1-\theta)v(e - P(m) - [1 - S(m)]L) + \theta v(e - P(m))$$

Plugging in $P(m) = (1-m)\cdot S(m)\cdot L$ (i.e. actuarially fair policy), we get

$$\begin{aligned} V(m,\theta) &= (1-\theta)v\Big(e - (1-m)\cdot S(m)\cdot L - [1-S(m)]L\Big) + \theta v\Big(e - (1-m)\cdot S(m)\cdot L\Big) \\ &= (1-\theta)v\Big(e - (1 - m\cdot S(m))L\Big) + \theta v\Big(e - (1-m)\cdot S(m)\cdot L\Big) \end{aligned}$$

Taking the first order conditions for incentive compatibility at $m = \theta$, we get

$$\begin{aligned} V_m(m,\theta) = (1-\theta)v'\Big(e - (1 - m\cdot S(m))L\Big)\Big(mS'(m)L + S(m)L\Big) \\ +\theta v'\Big(e - (1-m)\cdot S(m)\cdot L\Big)\Big(S(m)L - (1-m)S'(m)L\Big)\Big|_{m=\theta} = 0 \end{aligned}$$

$$\iff (1-\theta)v'\Big(e-(1-\theta S(\theta))L\Big)\Big(\theta S'(\theta)L+S(\theta)L\Big)+\theta v'\Big(e-(1-\theta)S(\theta)L\Big)\Big(S(\theta)L-(1-\theta)S'(\theta)L\Big) = 0.$$