Part I. Problems

Problem 1

(a)

The objective function is the expected payoff:

$$\begin{aligned} E[u(a)] = E[u(a,s)|a] &= E[1-(a-s)^2|a] \\ &= 1-a^2+2aE[s]-E[s^2] \\ &= 1-a^2+2aE[s]-Var[s]-E[s]^2. \end{aligned}$$

By the first order condition (take the derivative of the last expression with respect to a, set it =0 and solve for a), the optimal action is

$$a^* = E[s] = \frac{1}{2}.$$

The maximum expected payoff is

$$\begin{aligned} E[u(a^*)] &= 1-{a^*}^2+2a^*E[s]-E[s^2] \\ &= 1-E[s]^2+2E[s]E[s]-E[s^2] \\ &= 1-Var[s] \\ &= 1-\frac{1}{12} \\ &= \frac{11}{12}. \end{aligned}$$

(b)

Suppose s has an arbitrary CDF $F(s)$. As in part (a), the optimal action is $a^* = E[s]$. However, note that a is bounded by the interval [-10,20]. Therefore,

$$a^* = \begin{cases} 20 & \text{if E[s]} > 20 \\ E[s] = \int s dF(s) & \text{otherwise} \\ -10 & \text{if E[s]} < \text{-10.} \end{cases}$$

The maximum expected payoff at an interior optimum is

$$E[u(a^*)] = 1-Var[s]$$

.

Remark: the function $1-(a-s)^2$ is called the quadratic scoring rule, and is sometimes used to elicit beliefs. If you want a forecast of, say, next year's GDP, you could tell the forecaster that her payment will be proportional to this function (where $a =$ her forecast and $s =$ the official GDP when it is reported), and it would be in her interest to think carefully and report her subjective expectation.

Problem 2

(a)

Suppose I have a CARA utility function

$$u(x) = -e^{-ax}, \quad a > 0$$

and my certainty equivalent (CE) is \$900. Then, a should satisfy

$$u(CE) = E[u(x)]$$

$$u(900) = \frac{1}{2}u(1200) + \frac{1}{2}u(800)$$

$$-e^{-900a} = -\frac{1}{2}e^{-1200a} - \frac{1}{2}e^{-800a}.$$

This transcendental equation has no closed form solution, but it can be solved numerically, e.g., using a spreadsheet. The approximate solution (to $\frac{e^{-900a}}{\frac{1}{2}e^{-1200a}+\frac{1}{2}e^{-800a}} = 1$) is 0.006093... $\approx$ 0.0061.

(b)

The mean of lottery L is

$$\mu_L = \frac{1}{2}800 + \frac{1}{2}1200 = 1000,$$

and the variance is

$$\sigma_L^2 = \frac{1}{2}(800 - \mu_L)^2 + \frac{1}{2}(1200 - \mu_L)^2 = 200^2.$$

The mean and variance of lottery M = [\$900 with prob. 1] are

$$\mu_M = 900$$

$$\sigma_M^2 = 0.$$

Thus, c should satisfy

$$1000 - c * 200^2 = 900 - c * 0$$

$$\Longrightarrow c = \frac{1}{400} = 0.0025.$$

(c)

The mean and variance of lottery N = [\$1000 w/ prob. 0.001 and \$0 w/ prob. 0.999] are

$$\mu_N = 0.001 * 1000 = 1$$

$$\sigma_N^2 = 0.001 * (1000 - 1)^2 + 0.999 * (0 - 1)^2 \approx 999.$$

From part (b), the approximate mean-variance utility of N is

$$\mu_N - c\sigma_N^2 \approx 1 - 0.0025 * 999 = -1.4975 < 0.$$

From part (a), suppose $a = .0061$. The expected utility of lottery N is

$$Eu_N = 0.001 * (-e^{-.0061*1000}) + 0.999 * (-e^{-.0061*0}) \approx -.999.$$

(d)

Lottery O = [$0 w/ prob. 1] is first order stochastically dominated by lottery N. That is, whatever happens in lottery N, the result is either positive or $0, which is at least as good as $0. So common sense and FOSD say that you should prefer N to O. The CARA utility for O is -1 and for N it is a slightly larger number, so it also says that you should prefer N to O

However, the approximate mean-variance utility of lottery O is 0, which is higher than that of lottery N. It says you should prefer the certainty of $0 to the slight "risk" of a different outcome, even if that outcome is better!

This underlines the point in the class Notes that mean-variance is a good approximation when outcomes are tightly bunched around the mean or when the outcome is normally distributed (or when CARA is constant), but it can be a bad approximation in other cases.

Problem 3

(a)

The joint probability is computed as

$$\begin{aligned} P(d, t1, t2) &= P(t1, t2|d) * P(d) \\ &= P(t1|d) * P(t2|d) * P(d) \quad \text{(by conditional independence of } t1 \text{ and } t2\text{).} \end{aligned}$$

Table 1: Prior Probabilities and Likelihoods

Disease (d)	P(d)	P(t1=pos | d)	P(t1=neg | d)	P(t2=pos | d)	P(t2=neg | d)
A	0.6	0.7	0.3	0.2	0.8
B	0.4	0.4	0.6	0.5	0.5

Table 2: Joint Probabilities

Disease (d)	P(d, pos, pos)	P(d, pos, neg)	P(d, neg, pos)	P(d, neg, neg)
A	.6 * .7 * .2 = .084	.6 * .7 * .8 = .336	.6 * .3 * .2 = .036	.6 * .3 *.8 = .144
B	.4 * .4 * .5 = .08	.4 * .4 * .5 = .08	.4 * .6 *.5 = .12	.4 *.6 * .5 = .12

(b)

The prior probability can be computed as

$$P(t) = P(t, A) + P(t, B), \quad t = t1, t2$$

and the joint probability as

$$P(t, d) = P(t|d) * P(d), \quad t = t1, t2.$$

Table 3: Joint Probabilities

Disease (d)	P(t1=pos, d)	P(t1=neg, d)	P(t2=pos, d)	P(t2=neg, d)
A	.7 * .6 = .42	.3 * .6 = .18	.2 * .6 = .12	.8 * .6 = .48
B	.4 * .4 = .16	.6 * .4 = .24	.5 * .4 = .2	.5 * .4 = .2

Table 4: Prior Probabilities

P(t1=pos)	P(t1=neg)	P(t2=pos)	P(t2=neg)
.58	.42	.32	.68

(c)

For $t = t1, t2$, the posterior probability when only one test if performed can be computed as

$$\begin{aligned} P(d|t) &= \frac{P(t, d)}{P(t)} \\ &= \frac{P(t, d)}{P(t|A) * P(A) + P(t|B) * P(B)}. \end{aligned}$$

Table 5: Posterior Probabilities (one test)

d	P(d | t1=pos)	P(d | t1=neg)	P(d | t2=pos)	P(d | t2=neg)
A	.42 / (.42 + .16) ≈ .724	.18 / (.18 + .24) ≈ .429	.12 / (.12 + .2) = .375	.48 / (.48 + .2) ≈ .706
B	.16 / (.42 + .16) ≈ .276	.24 / (.18 + .24) ≈ .571	.2 / (.12 + .2) = .625	.2 / (.48 + .2) ≈ .294

(d)

The posterior probability when two tests are performed can be computed as

$$\begin{aligned} P(d|t1, t2) &= \frac{P(d, t1, t2)}{P(t1, t2)} \\ &= \frac{P(d, t1, t2)}{P(t1, t2|A) * P(A) + P(t1, t2|B) * P(B)} \\ &= \frac{P(d, t1, t2)}{P(t1|A) * P(t2|A) * P(A) + P(t1|B) * P(t2|B) * P(B)}. \end{aligned}$$

Table 6: Posterior Probabilities (two tests)

Disease (d)	P(d | pos, pos)	P(d | pos neg)	P(d | neg, pos)	P(d | neg, neg)
A	.512	.808	.231	.545
B	.488	.192	.769	.455

Problem 4

(a)

The expected payoff with imperfect information is

$$Eu = .1 * 20 + .5 * 5 + .4 * (-10) = .5.$$

because each type of salesman is hired. The expected payoff with perfect information is

$$Eu = .1 * 20 + .5 * 5 + 0 * (-10) = 4.5,$$

because under perfect information the bad type will not be hired. Therefore, the value of perfect information is

$$VPI = 4.5 - 0.5 = 4.$$

(b)

The likelihood (that a salesman would sell n cars in 4 days) is

$$P(n|type) = e^{-4\lambda} \frac{(4\lambda)^n}{n!}.$$

The posterior probability (of the type of an n-car-selling salesman) is

$$P(type|n) = \frac{P(n|type) * P(type)}{\sum_{type} P(n|type) * P(type)}$$

Table 7: Likelihoods and Posterior Probabilities

Type	P(n | type)	P(type | n)
Great	$\frac{2^n}{n!e^2}$	$\frac{0.1*\frac{2^n}{n!e^2}}{0.1*\frac{2^n}{n!e^2}+0.5*\frac{1}{n!e}+0.4*\frac{(1/2)^n}{n!e^{1/2}}}$
Good	$\frac{1}{n!e}$	$\frac{0.5*\frac{1}{n!e}}{0.1*\frac{2^n}{n!e^2}+0.5*\frac{1}{n!e}+0.4*\frac{(1/2)^n}{n!e^{1/2}}}$
Poor	$\frac{(1/2)^n}{n!e^{1/2}}$	$\frac{0.4*\frac{(1/2)^n}{n!e^{1/2}}}{0.1*\frac{2^n}{n!e^2}+0.5*\frac{1}{n!e}+0.4*\frac{(1/2)^n}{n!e^{1/2}}}$

The gross expected payoff of the dealer when he observes n cars sold in the last week is 0 if he doesn't hire, and is the conditional expected value if he hires:

$$Eu(n) = max\Big\{0, P(great|n) * 20 + P(good|n) * 5 + P(poor|n) * (-10)\Big\}.$$

Table 8 shows the dealer's hiring decision given n.

Table 8: Decision Table

n	0	1	2	3	4 ≤
$P(n\|great)$	0.1353	0.2707	0.2707	0.1804	0.1429
$P(n\|good)$	0.3679	0.3679	0.1839	0.0613	0.0190
$P(n\|poor)$	0.6065	0.3033	0.0758	0.0126	0.0018
$P(great\|n)$	0.0308	0.0815	0.1812	0.3357	0.5836
$P(good\|n)$	0.4180	0.5535	0.6157	0.5703	0.3878
$P(poor\|n)$	0.5513	0.3650	0.2030	0.0940	0.0286
$P(n)$	0.4401	0.3323	0.1494	0.0538	0.0245
$Eu(n\|hire)$	-2.808	0.746	4.673	8.625	13.325
Hiring Decision	**no**	**yes**	**yes**	**yes**	**yes**

The decision only changes when $n = 0$. Therefore the net value of information is

$$V_I = \underbrace{2.808}_{-Eu(0)} * \underbrace{0.4401}_{P(0)} - \underbrace{0.04}_{\text{hiring cost}} = 1.196.$$

(c)

The posterior probability (of the type of the salesman who sold 2 cars in the first week and n_2 cars in the second) is

$$\begin{aligned} P(type|n_1 = 2, n_2) &= \frac{P(type, 2, n_2)}{P(2, n_2)} \\ &= \frac{P(2|type) \cdot P(n_2|type) \cdot P(type)}{\sum_{type} P(2|type) \cdot P(n_2|type) \cdot P(type)}, \end{aligned}$$

since the joint probability can be computed as

$$\begin{aligned} P(type, n_1, n_2) &= P(n_1, n_2|type) \cdot P(type) \\ &= P(n_1|type) \cdot P(n_2|type) \cdot P(type). \end{aligned}$$

Table 9 displays the decision of the dealer given n_2.

Table 9: Decision Table (given $n_1 = 2$)

n_2	0	1	2	3	$4 \leq$
$P(n_1 \mid great)$	0.2707	0.2707	0.2707	0.2707	0.2707
$P(n_1 \mid good)$	0.1839	0.1839	0.1839	0.1839	0.1839
$P(n_1 \mid poor)$	0.0758	0.0758	0.0758	0.0758	0.0758
$P(n_2 \mid great)$	0.1353	0.2707	0.2707	0.1804	0.1429
$P(n_2 \mid good)$	0.3679	0.3679	0.1839	0.0613	0.0190
$P(n_2 \mid poor)$	0.6065	0.3033	0.0758	0.0126	0.0018
$P(great \mid n_1, n_2)$	0.0655	0.1455	0.2760	0.4478	0.6825
$P(good \mid n_1, n_2)$	0.6054	0.6719	0.6374	0.5170	0.3082
$P(poor \mid n_1, n_2)$	0.3291	0.1826	0.0866	0.0351	0.0094
$P(n1, n2)$	0.0559	0.0504	0.0265	0.0109	0.0057
$Eu(n1, n2)$	1.0465	4.4428	7.8409	11.1904	15.0961
Hiring Decision	**yes**	**yes**	**yes**	**yes**	**yes**

In this case, there is no change in decision from the message (i.e. the second week performance of the salesman). Therefore, as the value of information is 0 (which makes the net value -0.04), at the end of the first week the dealer will not extend the try-out period to the second week but will make the decision of hiring the salesman permanently.

Note: Alternatively, we can solve this problem in the exactly same process as in part (b) by only replacing the prior probabilities and get the same results as above.

Problem 5

The posterior probability can be computed as

$$P(new|prod) = \frac{P(prod|new) * P(new)}{P(prod|new) * P(new) + P(prod|old) * P(old)},$$

where *new* and *old* represents the states that my rival company has new technology or not, respectively, and *prod* indicates the level of reported productivity. With the given information, it can be calculated as

$$P(new|prod) \quad = \quad \frac{\frac{1}{\sqrt{2\pi}} exp\big(-\frac{(prod-12)^2}{2}\big) * 0.1}{\frac{1}{\sqrt{2\pi}} exp\big(-\frac{(prod-12)^2}{2}\big) * 0.1 + \frac{1}{\sqrt{2\pi}} exp\big(-\frac{(prod-10)^2}{2}\big) * 0.9}$$

As we need this to be greater than 0.5,

$$\frac{\frac{1}{\sqrt{2\pi}}exp\big(-\frac{(prod-12)^2}{2}\big)*0.1}{\frac{1}{\sqrt{2\pi}}exp\big(-\frac{(prod-12)^2}{2}\big)*0.1+\frac{1}{\sqrt{2\pi}}exp\big(-\frac{(prod-10)^2}{2}\big)*0.9} \geq 0.5$$

$$\frac{1}{\sqrt{2\pi}}exp\big(-\frac{(prod-12)^2}{2}\big)*0.1-\frac{1}{\sqrt{2\pi}}exp\big(-\frac{(prod-10)^2}{2}\big)*0.9 \geq 0$$

$$exp\bigg(-\frac{(prod-12)^2}{2}\bigg)\bigg/exp\bigg(-\frac{(prod-10)^2}{2}\bigg)-9 \geq 0$$

$$exp\bigg(-\frac{(prod-12)^2}{2}+\frac{(prod-10)^2}{2}\bigg)-9 \geq 0$$

$$exp\bigg(-\frac{44-4prod}{2}\bigg) \geq 9$$

$$2prod-22 \geq \ln 9$$

$$\therefore prod \geq 11+\frac{1}{2}\ln 9 \approx 12.1$$

Problem 6

(a)

There are two states:

$$ADEQUATE(A) \quad \text{and} \quad SUBSTANDARD(S)$$

and two actions:

$$adequate(a) \quad \text{and} \quad substandard(s).$$

The mean test outcome $\bar{X}$ of the test outcomes $\{X_1, X_2, ..., X_n\}$ is distributed as

$$\bar{X} \sim \begin{cases} N(1, 9/n) & \text{if adequate (A)} \\ N(-1, 9/n) & \text{if substandard (S).} \end{cases}$$

Using BI, we first consider the final choice, between adequate (a) and substandard (s) given the observed mean test outcome $\bar{X}$ for test size n. The expected payoff of a/s choice is

$$\begin{aligned} E[u(a|X,n)] &= -nP(A|\bar{X})-(n+1000)P(S|\bar{X}) \\ E[u(s|X,n)] &= -nP(S|\bar{X})-(n+1000)P(A|\bar{X}) \end{aligned}$$

Choosing adequate (a) is optimal, if

$$E[u(a|X,n)] = -nP(A|\bar{X})-(n+1000)P(S|\bar{X}) \geq -nP(S|\bar{X})-(n+1000)P(A|\bar{X}) = E[u(s|X,n)]$$

$$-n(P(A|\bar{X})+P(S|\bar{X}))-1000P(S|\bar{X}) \geq -n(P(S|\bar{X})+P(A|\bar{X}))-1000P(A|\bar{X})$$

$$P(S|\bar{X}) \geq P(A|\bar{X}),$$

which is, by Bayes' theorem, equivalent to

$$\frac{P(\bar{X}|S)P(S)}{P(\bar{X}|A)P(A)+P(\bar{X}|S)P(S)} \geq \frac{P(\bar{X}|A)P(A)}{P(\bar{X}|A)P(A)+P(\bar{X}|S)P(S)}$$

$$\Leftrightarrow P(\bar{X}|S) \geq P(\bar{X}|A) \quad \text{(given the prior probabilities).}$$

By using the pdf of Normal distribution, we get

$$exp\Big(-\frac{1}{2}\Big(\frac{\bar{X}-1}{3/\sqrt{n}}\Big)^2\Big) \geq exp\Big(-\frac{1}{2}\Big(\frac{\bar{X}+1}{3/\sqrt{n}}\Big)^2\Big)$$

$$(\bar{X}-1)^2 \leq (\bar{X}+1)^2$$

$$\bar{X} \geq 0.$$

Therefore, the decision rule is "choose a if $\bar{X} \geq 0$ and choose s otherwise."
(You might use symmetry to obtain this natural decision rule more quickly, but the long method used above shows how to generalize to non-symmetric priors and/or non-symmetric loss (or utility) functions.)

Continuing BI, we now consider the strategy of the choice of how many units to test. The expected payoff is

$$\begin{aligned} E[u|n] &= P(\bar{X} \geq 0)\Big(-nP(A|\bar{X} \geq 0) - (n+1000)P(S|\bar{X} \geq 0)\Big) \\ &\quad + P(\bar{X} < 0)\Big(-nP(S|\bar{X} < 0) - (n+1000)P(A|\bar{X} < 0)\Big) \end{aligned}$$

By Bayes' theorem,

$$P(A|\bar{X} \geq 0) = \frac{P(\bar{X} \geq 0|A)P(A)}{P(\bar{X} \geq 0|A)P(A) + P(\bar{X} \geq 0|S)P(S)}$$

$$P(A|\bar{X} < 0) = \frac{P(\bar{X} < 0|A)P(A)}{P(\bar{X} < 0|A)P(A) + P(\bar{X} < 0|S)P(S)}$$

For the ease of derivation,express the probability using cdf of standardized normal distribution.

$$P(\bar{X} \geq 0|A) = P\big(z \equiv \frac{\bar{X}-1}{3/\sqrt{n}} \geq \frac{-1}{3/\sqrt{n}}\big) = 1 - \Phi\big(\frac{-1}{3/\sqrt{n}}\big)$$

Similarly,

$$P(\bar{X} < 0|A) = \Phi(\frac{-1}{3/\sqrt{n}})$$
$$P(\bar{X} \geq 0|S) = 1 - \Phi(\frac{1}{3/\sqrt{n}})$$
$$P(\bar{X} < 0|S) = \Phi(\frac{1}{3/\sqrt{n}})$$

Since standard normal distribution is symmetric around 0,

$$P(\bar{X} \geq 0|A) = P(\bar{X} < 0|S)$$
$$P(\bar{X} < 0|A) = P(\bar{X} \geq 0|S)$$

Therefore, the maximal expected utility given N simplifies to

$$\begin{aligned} E[u|n] &= -n - 1000\Phi(\frac{-1}{3/\sqrt{n}}) \\ &= -n - 1000\int_{\infty}^{\frac{-1}{3/\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \end{aligned}$$

The agent choose n to maximize the expected utility. Using Leibnitz's rule, F.O.C is

$$\begin{aligned} \frac{dE[u|n]}{dn} = 0 &\iff -1 - 1000\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{-1}{3/\sqrt{n}}\right)^2}\left(-\frac{1}{6}n^{-\frac{1}{2}}\right) = 0 \\ &\iff \frac{500}{3\sqrt{2\pi}} \cdot e^{-\frac{n}{18}} \cdot n^{-\frac{1}{2}} = 1 \\ &\iff \ln\left(\frac{500}{3\sqrt{2\pi}}\right) - \frac{n}{18} - \frac{1}{2}\ln n = 0 \\ &\iff \frac{n}{9} + \ln n = 2\ln\left(\frac{500}{3\sqrt{2\pi}}\right). \end{aligned}$$

Solving numerically through a spreadsheet, we get the nearest integer to the solution as

$$n^* = 42.$$

(b)

The objective function is the same as before, to minimize expected loss. The only difference here is the prior probabilities. For extra credit, you can solve the problem as follows.

Given $\bar{X}$, P(A)=0.8, P(S)=0.2, and costs 1000 and 400, the agent choose a or s. As derived in part (a), the optimal decision is adequate(a) if

$$\frac{P(\bar{X}|S)P(S)}{400} \leq \frac{P(\bar{X}|A)P(A)}{1000}$$

$$\Leftrightarrow \frac{P(\bar{X}|A)}{P(\bar{X}|S)} \geq \frac{5}{8}$$

$$\bar{X} \geq -\frac{1}{2}(\ln 8 - \ln 5)\frac{9}{n} \equiv c(n)$$

$c(n)$ replace 0 in (a) and represents the indifferent point between choosing A and choosing S. The expected utility under this optimal strategy given n is

$$\begin{aligned} E[u|n] &= P(\bar{X} \geq c(n))\Big(- nP(A|\bar{X} \geq c(n)) - (n+1000)P(S|\bar{X} \geq 0)\Big) \\ &\quad + P(\bar{X} < c(n))\Big(- nP(S|\bar{X} < c(n)) - (n+400)P(A|\bar{X} < c(n))\Big) \end{aligned}$$

By Bayes' theorem,

$$\begin{aligned} P(A|\bar{X} \geq c(n)) &= \frac{P(\bar{X} \geq c(n)|A)P(A)}{P(\bar{X} \geq c(n)|A)P(A) + P(\bar{X} \geq c(n)|S)P(S)} \\ P(A|\bar{X} < c(n)) &= \frac{P(\bar{X} < c(n)|A)P(A)}{P(\bar{X} < c(n)|A)P(A) + P(\bar{X} < c(n)|S)P(S)} \end{aligned}$$

Thus, the expression of the expected utility simplifies to

$$\begin{aligned} E[u|n] &= -nP(\bar{X} \geq c(n)|A)P(A) + (-1000 - n)P(\bar{X} \geq c(n)|S)P(S) \\ &\quad -nP(\bar{X} < c(n)|S)P(S) + (-400 - n)P(\bar{X} < c(n)|A)P(A) \end{aligned}$$

For the ease of derivation,express the probability using cdf of standardized normal distribution.

$$P(\bar{X} \geq c(n)|A) = P\big(z \equiv \frac{\bar{X} - 1}{3/\sqrt{n}} \geq \frac{c(n) - 1}{3/\sqrt{n}}\big) = 1 - \Phi\big(\frac{c(n) - 1}{3/\sqrt{n}}\big)$$

Similarly,

$$\begin{aligned} P(\bar{X} < c(n)|A) &= \Phi\big(\frac{c(n) - 1}{3/\sqrt{n}}\big) \\ P(\bar{X} \geq c(n)|S) &= 1 - \Phi\big(\frac{c(n) + 1}{3/\sqrt{n}}\big) \\ P(\bar{X} < c(n)|S) &= \Phi\big(\frac{c(n) + 1}{3/\sqrt{n}}\big) \end{aligned}$$

Also, using the fact that

$$P(\bar{X} \geq c(n), A) + P(\bar{X} < c(n), A) + P(\bar{X} \geq c(n), S) + P(\bar{X} < c(n), S) = 1$$

The final expression of expected utility under the optimal strategy given n is

$$E[u|n] = -n - 1000\big(1 - \Phi\big(\frac{c(n) + 1}{3/\sqrt{n}}\big)\big)\frac{1}{5} - 400\Phi\big(\frac{c(n) - 1}{3/\sqrt{n}}\big)\frac{4}{5}$$

remind that $c(n) = -\frac{1}{2}(\ln 8 - \ln 5)\frac{9}{n}$, Similar to (a), the optimal n is the value which maximizes this expected utility.

$$n^* = 32$$

(c)

The sequential sampling problem takes more advanced techniques than those taught in 204b. It turns out that the optimal decision can be expressed in terms of two thresholds, say $U > 0$ and $L < 0$, for the posterior probability given the test results so far. U is defined by a Bellman equation expressing indifference between declaring a and continuing to sample, and L is defined by a similar indifference equation for declaring s or continuing to sample. You continue to sample until the posterior probability crosses one of the thresholds. Those thresholds are functions of the loss function parameters, the prior probabilities, and the cost of another test. You may encounter similar problems in "search theory" of unemployment, or in some theories of reaction time in decision making.

Problem 7

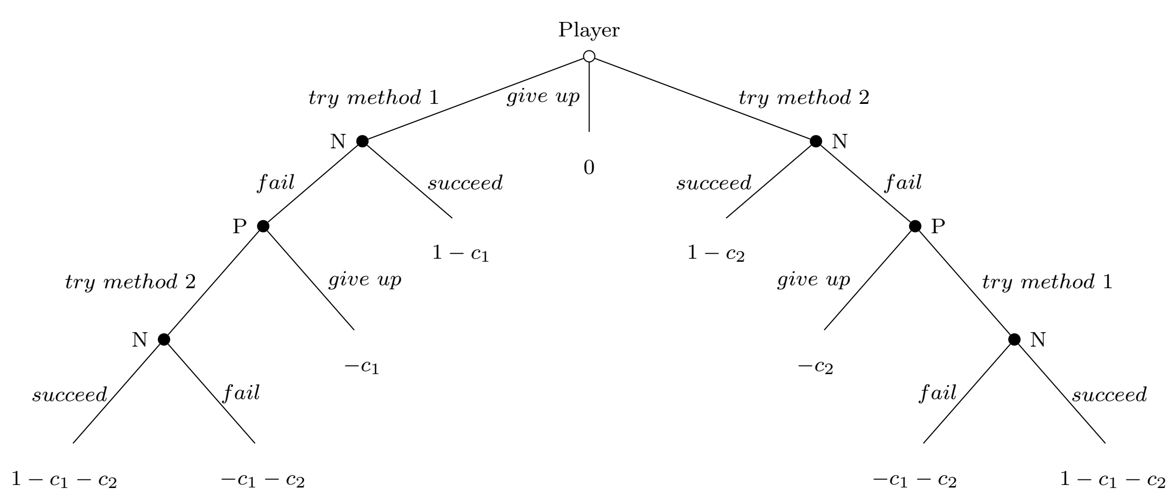
The optimal rule is "Sort methods in descending order of $\frac{p_i}{c_i}$. And stop whenever either (a) success is achieved or (b) all remaining methods have $\frac{p_i}{c_i} < 1$."

Part (b) holds because (as you can see from the decision tree) trying any remaining methods i changes payoff by $p_i - c_i < 0$ and hence reduces expected payoff. If (a) holds, of course, trying another method increase its cost with no gain. Hence the stopping rule is optimal.

To see that the descending order is optimal, suppose to the contrary that at some stage the plan calls for a lower benefit-cost ratio method k to be tried just before a higher benefit-cost ratio method k_{t+1}. We will show that switching the order increases expected payoff. This will establish the rule, since such k exists if and only if the rule is violated.

So, to complete the proof, it is sufficient to consider $K = 2$, with $1 < \frac{p_1}{c_1} < \frac{p_2}{c_2}$, and to show that it actually is better to try method 2 before method 1, i.e., to switch the order so that it is descending. Let

$$EV(1,2) \equiv \text{the expected payoff when trying the methods in the indicated order of 1 then 2}$$
$$EV(2,1) \equiv \text{the expected payoff when trying the methods in the reverse order of 2 then 1.}$$



From the decision tree, it is straightforward to see that $EV(1,2) < EV(2,1)$ if and only if

$$(p_1 - c_1) + (1 - p_1)(p_2 - c_2) < (p_2 - c_2) + (1 - p_2)(p_1 - c_1)$$

$$\Leftrightarrow \frac{p_1}{c_1} < \frac{p_2}{c_2}$$

Thus the proof is complete.

Part II. Textbook problems

6.C.1

We need to show that $q > \pi$ implies $D > \alpha^*$.
The decision maker's problem is

$$\max_{\alpha \geq 0} (1-\pi)u(w-\alpha q) + \pi u(w - \alpha q - D + \alpha).$$

The FOC (if $\alpha^* > 0$) is

$$-q(1-\pi)u'(w-\alpha^* q) + (1-q)\pi u'(w-\alpha^* q - D + \alpha^*) = 0$$

$$\frac{q(1-\pi)}{\pi(1-q)} = \frac{u'(w-\alpha^* q - D + \alpha^*)}{u'(w-\alpha^* q)}.$$

Suppose the insurance is not actuarially fair (i.e. $q > \pi$). Then

$$1 < \frac{q(1-\pi)}{\pi(1-q)} = \frac{u'(w-\alpha^* q - D + \alpha^*)}{u'(w-\alpha^* q)}$$

$$u'(w-\alpha^* q) < u'(w-\alpha^* q - D + \alpha^*).$$

Hence, given than the individual is risk-averse (i.e. $u'' < 0$), $D > \alpha^*$

6.C.2

(a)

Suppose that an individual has a Bernoulli utility function $u(\cdot)$ with the quadratic form

$$u(x) = \beta x^2 + \gamma x,$$

where $\beta < 0$ and $u(\cdot)$ has its upper bound at $-\gamma/2\beta$.
The expected utility is

$$\begin{aligned}
E[u] &= \int_{-\infty}^{-\gamma/2\beta} (\beta x^2 + \gamma x) dF(x) \\
&= \beta \int_{-\infty}^{-\gamma/2\beta} x^2 dF(x) + \gamma \int_{-\infty}^{-\gamma/2\beta} x dF(x) \\
&= \beta E[x^2] + \gamma E[x] \\
&= \gamma E[x] + \beta E[x]^2 + \beta Var[x] \\
&= \gamma(mean\, of\, F) + \beta(mean\, of\, F)^2 + \beta(variance\, of\, F)
\end{aligned}$$

(b)

We prove by contrast that $U(\cdot)$ is not compatible with any Bernoulli utility function. So suppose there is a Bernoulli utility function $u(\cdot)$ such that $U(F) = \int u(x)\, dF(x)$ for every distribution $F(\cdot)$. Let x and y be two amounts of money, $G(\cdot)$ be the distribution that puts probability one at x, and $H(\cdot)$ that puts probability one at y. Then

$$u(x) = u(G) = (mean\, of\, G) + (variance\, of\, G) = x - 0 = x$$

$$u(y) = u(H) = (mean\, of\, H) + (variance\, of\, H) = y - 0 = y$$

Thus $x \geq y$ if and only if $u(x) \geq u(y)$. Hence $u(\cdot)$ is strictly monotone. Now let $F_0(\cdot)$ be the distribution that puts one probability on 0 and $F(\cdot)$ be the distribution that puts $\frac{1}{2}$ on 0 and on $\frac{4}{r} > 0$.Since the mean and the variance of $F_0(\cdot)$ are 0, $U(F_0) = 0$. The strict monotonicity of $u(\cdot)$ this implies that $U(F) > 0$. However, $U(F) = \frac{2}{r} - r\frac{4}{r^2} = -\frac{2}{r} < 0$, which is a contradiction. Hence $U(\cdot)$is not compatible with any Bernoulli utility function.
An example with two lotteries with the property requested in the exercise was given in the above proof of incompatibility. (Note that if all we need to show were the incompatibility of $U(\cdot)$ and any Bernoulli utility function, the equality $u(x) = x$ obtained above would be sufficient to complete the proof, because this implies risk neutrality, which contradicts the fact that the variance of $F(\cdot)$ is subtracted in the definition of $U(\cdot)$)

6.C.18

(a)

- Arrow-Pratt coefficient of Absolute Risk Aversion

$$\begin{aligned} A(x) &= -\frac{u''(x)}{u'(x)} \\ &= -\frac{-\frac{1}{4}x^{-3/2}}{\frac{1}{2}x^{-1/2}} \\ &= \left.\frac{1}{2x}\right|_{x=5} = \frac{1}{10}. \end{aligned}$$

- Arrow-Pratt coefficient of Relative Risk Aversion

$$\begin{aligned} R(x) &= -\frac{xu''(x)}{u'(x)} \\ &= x\frac{1}{2x} = \frac{1}{2}. \end{aligned}$$

(b)

- Certainty Equivalent

$$\begin{aligned} CE &= u^{-1}(E[u]) \\ &= u^{-1}\left(\frac{1}{2}u(16) + \frac{1}{2}u(4)\right) \\ &= u^{-1}\left(\frac{1}{2}4 + \frac{1}{2}2\right) \\ &= u^{-1}(3) \\ &= 9. \end{aligned}$$

- Probability Premium

$$\begin{aligned} PP &= \frac{u(E[x]) - E[u(x)]}{u(x_{large}) - u(x_{small})} \\ &= \frac{u\left(\frac{1}{2}16 + \frac{1}{2}4\right) - E[\frac{1}{2}u(16) + \frac{1}{2}u(4)]}{u(16) - u(4)} \\ &= \frac{\sqrt{10} - 3}{2} \approx 0.08. \end{aligned}$$

(c)

- Certainty Equivalent

$$\begin{aligned} CE &= u^{-1}(E[u]) \\ &= u^{-1}\left(\frac{1}{2}u(36) + \frac{1}{2}u(16)\right) \\ &= u^{-1}\left(\frac{1}{2}6 + \frac{1}{2}4\right) \\ &= u^{-1}(5) \\ &= 25. \end{aligned}$$

$\Rightarrow$ The risk premium (i.e. $E[x] - CE$) is 1, the same as in part (b). Yet the expected wealth is higher at 25 compared to 10 before. So we can say that the individual became less risk averse as facing higher level of expected wealth. This is in accordance with the coefficient of absolute risk aversion, $A(x) = \frac{1}{2x}$, which is decreasing in x.

- Probability Premium

$$\begin{aligned} PP &= \frac{u(E[x]) - E[u(x)]}{u(x_{large}) - u(x_{small})} \\ &= \frac{u\left(\frac{1}{2}36 + \frac{1}{2}16\right) - E[\frac{1}{2}u(36) + \frac{1}{2}u(16)]}{u(36) - u(16)} \\ &= \frac{\sqrt{26} - 5}{2} \approx 0.05. \end{aligned}$$

⇒ The probability premium (PP) is, by definition, the excess in winning probability over fair odds that makes the individual indifferent between a certain outcome and a gamble. Geometrically, it is the height of the vertical "gap" between the utility function and the straight line, in relative terms of the whole range of utility values, and thus represents the curvature of the utility function. So, the lower PP, the less concave and the less risk-averse. This aligns with the interpretation above.

6.C.20

By definition, we have

$$u(CE) = \frac{1}{2}u(x+\varepsilon) + \frac{1}{2}u(x-\varepsilon).$$

Differentiating with respect to ε, we get

$$u'(CE)\frac{\partial CE}{\partial \varepsilon} = \frac{1}{2}u'(x+\varepsilon) - \frac{1}{2}u'(x-\varepsilon).$$

Again taking the derivative with respect to ε, we get

$$u''(CE)\left(\frac{\partial CE}{\partial \varepsilon}\right)^2 + u'(CE)\frac{\partial^2 CE}{\partial \varepsilon^2} = \frac{1}{2}u''(x+\varepsilon) + \frac{1}{2}u''(x-\varepsilon)$$

$$u''(CE)\left(\frac{\frac{1}{2}u'(x+\varepsilon) - \frac{1}{2}u'(x-\varepsilon)}{u'(CE)}\right)^2 + u'(CE)\frac{\partial^2 CE}{\partial \varepsilon^2} = \frac{1}{2}u''(x+\varepsilon) + \frac{1}{2}u''(x-\varepsilon)$$

$$\frac{\partial^2 CE}{\partial \varepsilon^2} = \left[\frac{1}{2}u''(x+\varepsilon) + \frac{1}{2}u''(x-\varepsilon) - u''(CE)\left(\frac{\frac{1}{2}u'(x+\varepsilon) - \frac{1}{2}u'(x-\varepsilon)}{u'(CE)}\right)^2\right]\Big/ u'(CE)$$

Taking the limit as $\varepsilon \to 0$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\partial^2 CE}{\partial \varepsilon^2} &= \lim_{\varepsilon\to 0}\left[\frac{1}{2}u''(x+\varepsilon) + \frac{1}{2}u''(x-\varepsilon) - u''(CE)\left(\frac{\frac{1}{2}u'(x+\varepsilon) - \frac{1}{2}u'(x-\varepsilon)}{u'(CE)}\right)^2\right]\Big/ u'(CE) \\
&= \lim_{\varepsilon\to 0}\left[\frac{1}{2}u''(x) + \frac{1}{2}u''(x) - u''(CE)\left(\frac{\frac{1}{2}u'(x) - \frac{1}{2}u'(x)}{u'(CE)}\right)^2\right]\Big/ u'(CE) \\
&= \lim_{\varepsilon\to 0}\frac{u''(x)}{u'(CE)} \\
&= \frac{u''(x)}{u'(x)} = -r_A(x),
\end{aligned}$$

where the second last inequality holds because $\lim_{\varepsilon\to 0} CE = x$.