1 Utility

CRRA Utility $u(x|r) = \frac{x^{1-r}}{1-r}$, $r \in (-\infty, \infty)$
CARA Utility $u(x|a) = 1 - e^{-ax}$, $a > 0$
Certainty Equivalent $u(CE) = \int u(x)\, dF(x)$
Risk Premium $u(\int x\, dF(x) - RP) = \int u(x)\, dF(x)$
Absolute Risk Aversion $A(x) = -\frac{u''(x)}{u'(x)}$
Relative Risk Aversion $R(x) = xA(x) = \frac{-xu''(x)}{u'(x)}$
Mean-Variance Approximation
$u(\overline{x} + h) = u(\overline{x}) + (x - \overline{x})u'(\overline{x}) + \frac{1}{2}(x - \overline{x})^2 u''(\overline{x}) + R^3$
$Eu = u(\overline{x}) - \frac{1}{2}A(\overline{x})\sigma_L^2 + R^3$
First Order Stochastic Dominance
$F(x) \geq G(x)\ \forall x$
Second Order Stochastic Dominance
$\mu_F = \mu_G$ & $\int_{-\infty}^{x} F(t)dt \geq \int_{-\infty}^{x} G(t)dt\ \forall x$

2 Bayes' Theorem

Basic Definitions
$p(s) = \sum_{z \in Z} p(s,z)$ (prior prob. of state s)
$p(z) = \sum_{s \in S} p(s,z)$ (message prob.)
$p(z|s) = \frac{p(s,z)}{p(s)}$ (likelihood)
$p(s|z) = \frac{p(s,z)}{p(z)}$ (posterior prob.)

Bayes theorem

(i) $p(s|z) = \frac{p(z|s)p(s)}{p(z)}$

(ii) $p(s|z) = \frac{p(z|s)p(s)}{\sum_{t \in S} p(z|t)p(t)}$

(iii) $\frac{p(s|z)}{p(t|z)} = \frac{p(z|s)p(s)}{p(z|t)p(t)}$

(iv) $\ln \frac{p(s|z)}{p(t|z)} = \ln \frac{p(z|s)}{p(z|t)} + \ln \frac{p(s)}{p(t)}$

Value of information
$V_I = \sum_{z \in Z} p(z) \sum_{s \in S} p(s|z)[u_z^*(s) - u_0^*(s)]$

3 Normal Form Games

Cookbook for NFG solutions
(i) Get NFG from story or EFG (should be a complete contingency plan)
(ii) Eliminate strictly dominated strategies (never-best-response are the candidates) and reduce the game. If only one profile remains, it is DS solution
(iii) Iterate step(i) until no more dominated strategies, if only one profile remains, it is IDDS
(iv) Inspect for mutual BR $\longrightarrow$ These are pure NE
(v) Check for mixed NE, $\sigma_i \in B_i(\sigma_{-i})$: for each $|subset| \geq 2$ of remaining pure strategies for each player, solve the set of simultaneous equations

$$f_1(s_1, \sigma_{-1}) = f_1(s_2, \sigma_{-1})$$
$$f_2(t_1, \sigma_{-2}) = f_2(t_2, \sigma_{-2})$$

Payoff function of mixed strategies (2x2)

$$f_1(\sigma_1, \sigma_{-1}) = \sum_{i=1}^{2} p_i \sum_{j=1}^{2} q_j f_1(s_i, t_j)$$

$where\, \sigma_1 = p_1 s_1 + (1-p_1)s_2, \sigma_{-1} = q_1 t_1 + (1-q_1)t_2$

4 Extensive Form Games

Cookbook for games of perfect information
(i) Convert each penultimate node ν into a terminal node: If ν is owned by player i, then use the branch with max payoff for i. If ν is owned by nature, then take expectation over payoff vectors
(ii) Iterate step 1 until you reach the initial node
(iii) Reconstruct each player's strategy for her choices in steps 1-2
(iv) The resulting profile is a SPNE.
(v) (For imperfect info) Find smallest subgames and their NE. Replace initial node of each subgame by (one of) its NE payoff vector. Iterate to a solution $\longrightarrow$ get one SPNE. Then iterate using other subgame NE (if any) to get all other SPNE.

5 BNE, PBE and Seq EQ

(i) Beliefs μ_i at each info set of i are consistent with common prior and likelihood from s_{-i}^* via Bayes

(ii) At each info set player i of realized type $\bar{\theta}_i$ maximizes $E(u_i|\mu_i, \bar{\theta}_i)$,$\forall\ s_i \epsilon S_i$, so

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s_i'(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i]$$

(iii) conditions (i, ii) hold in every subgame

(iv) solution is robust to sufficiently small trembles

(i) and (ii) constitute a Bayesian NE
(i) thru (iii) constitute a Perfect Bayesian NE
(i) thru (iv) constitute a sequential equilibrium

6 Repeated Games

Let the stage game be PD.
T finite: Only stage game NE are equilibria of the repeated game, i.e., always-defect is the unique NE.
T infinite: Cooperation can be sustained via trigger strategies as a NE of the repeated game if $d \geq d^*$ (discount factor).
Folk Thm: Any stage game feasible payoff vector that Pareto dominates a NE payoff is achievable as average payoff in a SPNE of the infinitely repeated game (via NE reversion strategies) if players are sufficiently patient.

7 Evolutionary Games

For payoff matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, let $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$ and $p^* = \frac{a_2}{a_1 + a_2}$. Then A is
HD type if $a_1, a_2 < 0$. Then $p^* \in (0,1)$ is a downcrossing, so it is the unique NE and EE: it is globally stable.
CO type if $a_1, a_2 > 0$. Then $p^* \in (0,1)$ is an upcrossing, so it is an unstable NE that separates the basins of attraction of the two pure strategy NE (also EE).

DS type if a_1 and a_2 have opposite signs. Then $p* \notin (0,1)$ so there is no mixed NE. The first pure strategy is dominant if $a_1 < 0 < a_2$ and the second is dominant if $a_1 > 0 > a_2$. Evolutionary dynamics always push the state towards a dominant strategy from any initial condition.
Replicator dynamics: equate the growth rate of each strategy share to its relative payoff.
$\dot{s}_i/s_i = w_i - \bar{w} = w_i \sum_j s_j - \sum_j w_j s_j = \sum_j (w_i - w_j) s_j$

8 Bargaining and Cooperative

NBS: Allocation which maximizes the product of players utility gains relative to a threat point.
Characteristic Function: Cooperative games are defined by a (superadditive) characteristic function that specifies the worth $v(K) \in R$ of each coalition $K \subset N$.
Convex game: $v(X) + V(Y) \leq v(X \cap Y) + v(X \cup Y)$
Core: Coalition K blocks allocation u if $\sum_{i \in K} \ u_i < v(K)$. That means they can do better by themselves. Core is all allocations unblocked by any $K \subset N$.
Shapley Value: SV is based on marginal contribution of each player to every K. The formula is, $\phi_i(v) = \frac{1}{n!} \sum_\rho MC_i(\rho)$, where ρ is a permutation of {1,...,n}.

9 Imperfect Competition

Monopolist's FOC: $q_m[p'(q_m)] + p(q_m) = c'(q_m)$
DWL: $dwl = \int_{q_m}^{q_0} [p(z) - c'(z)]\, dz$
Bertrand: Firms simultaneously choose price to maximize profit: $\pi_j(p_j, p_k) = x_j(p_j, p_k)[p_j - c]$. The unique NE is $p_j = p_k = c, \pi_j = \pi_k = 0$.
Cournot: Firms simultaneously choose quantity to maximize profit: $\pi_j(q_j, q_k) = P(q_j + q_k)q_j - cq_j$. The FOC is: $P'(q_j + q_k)q_j + P(q_j + q_k) = c$. NE is $q_j = q_k, \pi_j = \pi_k$. The equilibrium price is between p_m and p_0.
Hotelling: In the duopoly where firms choose location but not price and $p_1 = p_2 = p > c = c_1 = c_2$. The unique NE is for both firms to locate at middle point.

10 Adverse selection, Signalling, Screening

Adverse Selection in Lemons model: Seller knows quality $\theta = value\ to\ buyer$. Seller values at $r(\theta)$. $\Theta(p) = \theta : r(p) \leq p$ is the subset of sellers willing to sell at price p. Here a competitive eqm. is p^*, Θ^* s.t. $p^* = E(\theta | \theta \in \Theta^*)$ and $\Theta^* = \theta : r(p) \leq p^*$
Signaling: N first chooses $\theta \in \Theta$; then Informed player ("(sender") sends message $m(\theta)$. Then Uninformed player ("(receiver") picks action $a(m)$ after forming beliefs $\mu(\theta|m)$. PBE is $[m^*(\theta), a(\theta), \mu(\theta|m)]$ s.t.
1. $m^* \in argmax\ u_s(m,\ a^*(m),\ \theta)\ \forall\theta$
2. $a^*(m) \in argmax\ u_r(a)$ (pick a max Expected payoff)
3. $\mu(\theta|m)$ is consistent with Bayes given priors and $m^*(\theta)$

Screening: U-N-I, usually uninformed players offered menu to informed players. For example, buyers offer deferred contingent payment; self-selection of insurance customers to reveals some private information regarding riskiness.

11 P/A model

Effort observable: The Principal solves:
$\min_{w(\pi)_{\pi \in \Pi}} \int_{\underline{\pi}}^{\overline{\pi}} w(\pi) f(\pi/e) d\pi$ s.t. [PC]
where [PC] is: $E(U_A) = \int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi/e) d\pi - g(e) \geq \overline{u}$
For A: $U_A(w,e) = u(w) - g(e)$, with outside option $\overline{u}$.
The solution is $w_e^* = u^{-1}(\overline{u} + g(e))$ for any given e, and Principal max's net profit over all e.
Effort not observable, Agent risk-neutral:
This means $u''(w) = 0$, we set $u(w) = w$
Guess: $w^* = \pi - \alpha$. Check the Principal gets
$\max_{e \in e_H, e_L} \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi/e) d\pi - (\overline{u} + g(e))$
The Agents expected utility is:
$E(U_A)(w^*) = \max_{e \in e_H, e_L} \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi/e) d\pi - (\alpha + g(e))$
The expected payoffs to both P and A are the same as in Case 1.
$E(U_A) = \overline{u}$ and $\alpha = \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi/e^*) d\pi - (\overline{u} + g(e^*))$
where e^* is the efficient effort level.
Effort not observable, Agent risk-averse:
The Principal solves, for each $e \in \{e_H, e_L\}$
$\min E(w) = \int_{\underline{\pi}}^{\overline{\pi}} w(\pi) f(\pi/e) d\pi$ s.t. [PC]and [IC]
where [PC] is: $E(U_A) = \int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi/e) d\pi - g(e) \geq \overline{u}$
where [IC] is:
$\int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi/e) d\pi - g(e) \geq \int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi/\tilde{e}) d\pi - g(\tilde{e})$
FOC w.r.t $w(\pi)$ for $e = e_H$ is:

$$\frac{1}{u'(w(\pi))} = \gamma + \mu[1 - \frac{f(\pi/e_L)}{f(\pi/e_H)}]$$