Solution to Problem Set 7

1. The utility function is given by

$$U(w) = -e^{-w}.$$

Then the expected utility of the gamble is

$$pU(w_1) + (1-p)U(w_2) = p(-e^{-w_1}) + (1-p)(-e^{-w_2})$$

Let $w = x$ be the value we are interested in. Then,

$$p(-e^{-w_1}) + (1-p)(-e^{-w_2}) = -e^{-x}.$$

It follows that

$$-\ln(p(e^{-w_1}) + (1-p)(e^{-w_2})) = x.$$

2. Recall the agent has only two options: put all his wealth in one asset, or divide it among the two assets.

a. The expected utility of investing a fraction α in the first asset and $(1-\alpha)$ in the second one is given by

$$\int\int u(\alpha w(r_1+1) + (1-\alpha)w(r_2+1))dF(r_1)dF(r_2)$$

In addition,

$$\begin{aligned}
& \int\int u(\alpha w(r_1+1) + (1-\alpha)w(r_2+1))dF(r_1)dF(r_2) \\
\geq & \int\int \alpha u(w(r_1+1)) + (1-\alpha)u(w(r_2+1))dF(r_1)dF(r_2) \\
= & \alpha \int u(w(r_1+1))dF(r_1) + (1-\alpha)\int u(w(r_2+1))dF(r_2) \\
= & \int u(w(r_1+1))dF(r_1) = \int u(w(r_2+1))dF(r_2)
\end{aligned}$$

where the inequality follows by concavity of u and the last equality as r_1 and r_2 are identically distributed. Thus, the agent prefers to divide his wealth among the two assets.

b. The expected utility of investing a fraction α in the first asset and $(1-\alpha)$ in the second one is given by

$$\int\int u(\alpha w(r_1+1) + (1-\alpha)w(r_2+1))dF(r_1)dF(r_2)$$

In addition,

$$\begin{aligned}
& \int\int u(\alpha w(r_1+1)+(1-\alpha)w(r_2+1))dF(r_1)dF(r_2) \\
\leq & \int\int \alpha u(w(r_1+1))+(1-\alpha)u(w(r_2+1))dF(r_1)dF(r_2) \\
= & \ \alpha\int u(w(r_1+1))dF(r_1)+(1-\alpha)\int u(w(r_2+1))dF(r_2) \\
= & \int u(w(r_1+1))dF(r_1)=\int u(w(r_2+1))dF(r_2)
\end{aligned}$$

where the inequality follows by convexity of u and the last equality as r_1 and r_2 are also identically distributed. Thus, the agent prefers to invest all his money in a single asset.

3. First note that $\Pr\big(\text{first head occurs at } j^{th} \text{ flip}\big)$ is given by

$$\Pr\big(\text{outcome is tail for all first } j^{th}-1 \text{ flips}\big)\Pr\big(\text{outcome is tail at } j^{th} \text{ flip}\big)$$

Then,

$$\Pr(2^j)=(1-p)^{j-1}p.$$

a. The expected value of the bet is

$$EV=\sum_{j=1}^{\infty}2^j(1-p)^{j-1}p.$$

Then, if $p=1/2$

$$EV=\sum_{j=1}^{\infty}2^j(\frac{1}{2})^{j-1}(\frac{1}{2})=\sum_{j=1}^{\infty}1=\infty$$

b and c. The expected utility of the bet is

$$\begin{aligned}
EU &= \sum_{j=1}^{\infty}\ln(2^j)(1-p)^{j-1}p=\sum_{j=1}^{\infty}j\ln(2)(1-p)^{j-1}p \\
&= \ln(2)\underbrace{\sum_{j=1}^{\infty}j(1-p)^{j-1}p}_{\text{expected value of a geometric random variable}}=\ln(2)\frac{1}{p}
\end{aligned}$$

Then, if $p=1/2$

$$EU=2\ln(2).$$

d. Recall $\overline{w}$ is the amount of money that would give you the same utility as playing the game. Then,

$$\ln(\overline{w})=2\ln(2)=\ln(2^2)=\ln 4\Rightarrow\overline{w}=4.$$