Solution to Problem Set 6

1. For each part of this question it is convenient to write the objective function in a specific way.

a. The problem of the firm can be written as follows

$$\max_{y\in[0,\infty)} \{\pi(y) = py - C(y, w_1, w_2)\}$$

where $C(y, w_1, w_2)$ is the minimum cost function. Notice that $C(y, w_1, w_2)$ does not depend on p. In addition,

i) $y \in [0, \infty)$ (the feasible set does not depend on p)

ii) $\frac{\partial^2 \pi}{\partial y \partial p} = 1 > 0$

Then, the extremal selections of y^* are increasing in p.

b. The problem of the firm can be written as follows

$$\max_{x_1\in[0,\infty), y\in[0,\infty)} \{\pi(x_1, y) = py - w_1 x_1 - w_2 h(x_1, y)\}.$$

We want to find conditions such that the extremal selections of x_1^* decrease in p. To use Topkis, we consider $-x_1$ and provide conditions so that the extremal selections of $(-x_1^*, y^*)$ decrease in p.
We have that

i) $x_1 \in (-\infty, 0]$ and $y \in [0, \infty)$ (the feasible sets do not depend on p)

ii) $\frac{\partial^2 \pi}{\partial(-x_1)\partial y} = w_2 \frac{\partial^2 h(x_1,y)}{\partial x_1 \partial y}$

iii) $\frac{\partial^2 \pi}{\partial(-x_1)\partial p} = 0$, $\frac{\partial^2 \pi}{\partial p \partial y} = 1$

We can thereby apply Topki's theorem if $\frac{\partial^2 h(x_1,y)}{\partial x_1 \partial y} \geq 0$.

The following steps will prove that $\frac{\partial^2 h(x_1,y)}{\partial x_1 \partial y} = \frac{(f_1 f_{22} - f_2 f_{21})}{(f_2)^3}$.

Notice that

$$\frac{(f_1 f_{22} - f_2 f_{21})}{(f_2)^2} = -\frac{\partial MRTS_{12}}{\partial x_2}$$

then $\frac{(f_1 f_{22} - f_2 f_{21})}{(f_2)^2} \geq 0$ holds if and only if $\frac{\partial(f_1/f_2)}{\partial x_2} \leq 0$, which means that x_1 is an inferior input. This result is similar to the one of the consumer when we showed that if $\frac{\partial MRS}{x_2} \leq 0$ holds then x_1 is an inferior good.

To show our claim, notice that

$$y = f(x_1, h(x_1, y)). \tag{1}$$

Differentiating (1) with respect to y, we get

$$1 = f_2 \frac{\partial h(x_1, y)}{\partial y} \Rightarrow \frac{\partial h(x_1, y)}{\partial y} = \frac{1}{f_2}.$$

Differentiating (1) with respect to x_1, we get

$$0 = f_1 + f_2 \frac{\partial h(x_1, y)}{\partial x_1} \Rightarrow \frac{\partial h(x_1, y)}{\partial x_1} = \frac{-f_1}{f_2}.$$

Finally, taking the cross partial derivative of (1) with respect to x_1 and y

$$\frac{\partial h(x_1, y)}{\partial y}(f_{21} + f_{22} \frac{\partial h(x_1, y)}{\partial x_1}) + f_2 \frac{\partial^2 h(x_1, y)}{\partial x_1 \partial y} = 0.$$

Making the corresponding substitutions we obtain

$$\frac{\partial^2 h(x_1, y)}{\partial x_1 \partial y} = -\frac{1}{(f_2)^2}(f_{21} - \frac{f_{22} f_1}{f_2}) = \frac{(f_1 f_{22} - f_2 f_{21})}{(f_2)^3}.$$

Then, the extremal selections of x_1^* decrease in p if x_1 is an inferior input.

c. The problem of the firm can be written as follows

$$\max_{x_1}\{\max_{x_2}[pf(x_1, x_2) - w_1 x_1 - w_2 x_2]\} = \max_{x_1}\{\max_{x_2} F(x_1, x_2)\} = \max_{x_1} F(x_1, x_2^*(w_2, p, x_1)).$$

The key advantage of this reformulation of the problem is that $x_2^*(w_2, p, x_1)$ does not depend on w_1!

In addition,

i) $x_1 \in [0, \infty)$ (the feasible set does not depend on w_1)

ii) $\frac{\partial F(x_1, x_2^*(w_2, p, x_1))}{\partial x_1 \partial - w_1} = 1 \geq 0$

Then, the extremal selections of x_1^* are decrease in w_1.

2. Consider the budget constraint

$$p_1 x_1 + p_2 z = y.$$

Thus,

$$x_1 = \frac{y - p_2 z}{p_1}$$

The problem of the consumer can be written as

$$\max_{z \in \left[0, \frac{y}{p_2}\right]} U\left(\frac{y - p_2 z}{p_1}, f(z)\right)$$

In addition,

i) $z \in \left[0, \frac{y}{p_2}\right]$ (the feasible set is ascending in y)

ii) $\frac{\partial U}{\partial y} = \frac{1}{p_1}\left(-\frac{p_2}{p_1}U_{11} + U_{12}f'\right)$

Condition ii) is positive if

$$-p_2 U_{11} + p_1 U_{12} f' \geq 0.$$

Then, the extremal selections of x_2^* increase in y if $-p_2 U_{11} + p_1 U_{12} f' \geq 0$.