Solution to Problem Set 5

1. The objective function is given by

$$\pi(x,y) = \alpha tx + ty - x^2 - y^2 - xy, \qquad t \in [0,\infty),\ \alpha \in (-\infty,\infty).$$

Differentiating this function with respect to x and y we get

$$\frac{\partial \pi}{\partial x} = \alpha t - 2x - y \text{ and } \frac{\partial \pi}{\partial y} = t - 2y - x.$$

a. We assume y is fixed.

Thus, the optimal solution is $x^* = (\alpha t - y)/2$. Notice that $\partial x^*/\partial t = \alpha/2$. It follows that x^* is increasing in t if $\alpha \geq 0$ and it is decreasing in t if $\alpha \leq 0$.

In addition,

i) $x \in [0,\infty)$ (the feasible set does not depend on t)

ii) $\partial^2\pi/\partial x\partial t = \alpha$

Then, since the solution is unique, we get the same result by using Topkis' Theorem.

b. We assume x is fixed.

Thus, the optimal solution is $y^* = (t - x)/2$. Notice that $\partial y^*/\partial t = 1/2$. It follows that y^* is increasing in t. In this case, α does not play any role.

In addition,

i) $y \in [0,\infty)$ (the feasible set does not depend on t)

ii) $\partial^2\pi/\partial x\partial t = 1$

Then, since the solution is unique, we get the same result by using Topkis' Theorem.

c. In this case, the optimal solution is

$$x^* = \frac{(2\alpha - 1)}{3}t \text{ and } y^* = \frac{(2-\alpha)}{3}t.$$

It follows that

If	$\alpha \in (-\infty, \frac{1}{2})$	then	x^* is decreasing in t and y^* is increasing in t
If	$\alpha \in [\frac{1}{2}, 2]$	then	x^* and y^* are both increasing in t
If	$\alpha \in (2,\infty)$	then	x^* is increasing in t and y^* is decreasing in t

In this case, we cannot apply —at least directly— Topkis' Theorem as $\partial^2\pi/\partial x\partial y = -1 < 0$.

d. (I already interpreted the results in terms of Topkis' Theorem.)

2. The objective function is given by

$$\pi(q, m, e) = qP(q; m) - \frac{cq}{e} - re - h(m; \theta).$$

a. Let $R = qP(q; m)$. Then,

$$\frac{\partial^2 R}{\partial q \partial m} = \frac{\partial P}{\partial m} + q\frac{\partial^2 P}{\partial q \partial m} \geq 0.$$

The sufficient conditions are: (i) inverse demand increases in m ($\frac{\partial P}{\partial m} \geq 0$); and (ii) the marginal effect of marketing on the inverse demand increases with q ($\frac{\partial^2 P}{\partial q \partial m} \geq 0$).

b. We assume e is fixed.

Comparative statics with respect to θ (let us consider $-\theta$ instead)

i) $q \in [0, \infty)$ and $m \in [0, \infty)$ do not depend on $-\theta$

ii) $\frac{\partial^2 \pi}{\partial q \partial m} = \frac{\partial P}{\partial m} + q\frac{\partial^2 P}{\partial q \partial m} = \frac{\partial^2 R}{\partial q \partial m} \geq 0$ (by assumption —see part a)

iii) $\frac{\partial^2 \pi}{\partial q \partial (-\theta)} = 0$ and $\frac{\partial^2 \pi}{\partial m \partial (-\theta)} = \frac{\partial^2 h}{\partial m \partial \theta} \geq 0$ (since h is supermodular by assumption)

Then, the extremal selections of q^* and m^* are increasing in $-\theta$ which means they are decreasing in θ.

Comparative statics with respect to e

i) $q \in [0, \infty)$ and $m \in [0, \infty)$ do not depend on e

ii) $\frac{\partial^2 \pi}{\partial q \partial m} = \frac{\partial P}{\partial m} + q\frac{\partial^2 P}{\partial q \partial m} = \frac{\partial^2 R}{\partial q \partial m} \geq 0$ (by assumption —see part a)

iii) $\frac{\partial^2 \pi}{\partial q \partial e} = \frac{c}{e^2} \geq 0$ and $\frac{\partial^2 \pi}{\partial m \partial e} = 0$

Then, the extremal selections of q^* and m^* are increasing in e.

c. In the long run none of the choice variables are fixed. Then,

i) $q \in [0, \infty)$, $m \in [0, \infty)$, and $e \in [0, \infty)$ do not depend on $-\theta$

ii) $\frac{\partial^2 \pi}{\partial q \partial m} = \frac{\partial P}{\partial m} + q\frac{\partial^2 P}{\partial q \partial m} = \frac{\partial^2 R}{\partial q \partial m} \geq 0$ (by assumption —see part a)
$\frac{\partial^2 \pi}{\partial q \partial e} = \frac{c}{e^2} \geq 0$
$\frac{\partial^2 \pi}{\partial m \partial e} = 0$

iii) $\frac{\partial^2 \pi}{\partial q \partial(-\theta)} = 0$

$\frac{\partial^2 \pi}{\partial m \partial(-\theta)} = \frac{\partial^2 h}{\partial m \partial \theta} \geq 0$ (since h is supermodular by assumption)

$\frac{\partial^2 \pi}{\partial e \partial(-\theta)} = 0$

Then, the extremal selections of e^* are increasing in $-\theta$ which means they are decreasing in θ.

d. The effect is greater in the long run. For formal proof refer to Milgrom and Roberts 1995b.

e. By Envelope theorem,

$$\frac{\partial \pi^*}{\partial r} = -e^* \leq 0.$$

In addition, by part c, we get

$$\frac{\partial^2 \pi^*}{\partial r \partial \theta} = -\frac{\partial e^*}{\partial \theta} \geq 0.$$