Solution to Problem Set 4

1. The problem of the firm is given by

$$\min_{x\in\mathbb{R}^2_+}\{w_1x_1 + w_2x_2 : y = \min\{\alpha x_1, \beta x_2\}\}$$

Then, at the optimum, we must have $\alpha x_1^* = \beta x_2^* = y$.
It follows that

$$x_1^* = \frac{y}{\alpha} \text{ and } x_2^* = \frac{y}{\beta}.$$

Then,

$$C(w,y) = \frac{y}{\alpha}w_1 + \frac{y}{\beta}w_2 = y\left(\frac{1}{\alpha}w_1 + \frac{1}{\beta}w_2\right)$$

2. Notice that

$$f(x_1(w_1,w_2,y), x_2(w_1,w_2,y)) = y.$$

Differentiating this expression with respect to w_2 we det

$$f_1\frac{\partial x_1(w_1,w_2,y)}{\partial w_2} + f_2\frac{\partial x_2(w_1,w_2,y)}{\partial w_2} = 0.$$

We know that $f_1 > 0$, $f_2 > 0$, and $\frac{\partial x_2(w_1,w_2,y)}{\partial w_2} \leq 0$. Thus, we must have $\frac{\partial x_1(w_1,w_2,y)}{\partial w_2} \geq 0$.

3. (a. and b. together) The objective function is given by

$$\pi = pf(x_1,x_2) - w_1x_1 - w_2x_2.$$

The first order conditions are given by

$$\begin{aligned} pf_1(x_1^*,x_2^*) - w_1x_1^* &= 0 \\ pf_2(x_1^*,x_2^*) - w_2x_2^* &= 0 \end{aligned}$$

Differentiating with respect to w_1, we get

$$\begin{aligned} pf_{11}\frac{\partial x_1^*}{\partial w_1} - pf_{12}\frac{\partial x_2^*}{\partial w_1} &= 1 \\ pf_{21}\frac{\partial x_1^*}{\partial w_1} - pf_{22}\frac{\partial x_2^*}{\partial w_1} &= 0 \end{aligned}$$

By Crammer's rule,

$$\frac{\partial x_1^*}{\partial w_1} = \frac{\overbrace{\begin{vmatrix} 1 & pf_{12} \\ 0 & pf_{22} \end{vmatrix}}^{pf_{22}\leq 0 \text{ by concavity of } f}}{\underbrace{\begin{vmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{vmatrix}}_{\geq 0 \text{ by concavity of } f\text{. We assume } \neq 0}} \leq 0$$

$$\frac{\partial x_2^*}{\partial w_1} = \frac{\overbrace{\begin{vmatrix} pf_{11} & 1 \\ pf_{21} & 0 \end{vmatrix}}^{-pf_{21}}}{\underbrace{\begin{vmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{vmatrix}}_{\geq 0 \text{ by concavity of } f. \text{ We assume } \neq 0}} \Rightarrow sgn(\frac{\partial x_2^*}{\partial w_1}) = sgn(-pf_{21}).$$

It follows that

$$\frac{\partial x_2^*}{\partial w_1} \begin{cases} \geq 0 & \text{if} \quad f_{21} \leq 0 \\ \leq 0 & \text{if} \quad f_{21} \geq 0 \end{cases}.$$

Notice that $f_{21} \geq 0$ means that inputs are complements to each other.

4. Since f is strictly concave, $\forall t > 1$

$$f\left(\frac{1}{t}x + \left(1 - \frac{1}{t}\right)x'\right) > \frac{1}{t}f(x) + \left(1 - \frac{1}{t}\right)f(x') \qquad \forall x, x' \in \mathbb{R}_+^n$$

Let $x' = 0$, then

$$f\left(\frac{1}{t}x\right) > \frac{1}{t}f(x) \qquad \forall x \in \mathbb{R}_+^n.$$

Let $\frac{1}{t}x = x''$, so that $x = tx''$.Then

$$tf(x'') > f(tx'') \qquad \forall x'' \in \mathbb{R}_+^n \text{ and } \forall t > 1$$

which means that f has decreasing returns to scale.