Solution to Problem Set 3

1. The Lagrangian for the problem is given by

$$L = U(H, kC) - \lambda(p_h H + p_c C - y)$$

a. The first order conditions are given by

$$\begin{aligned}
\frac{\partial L}{\partial H} &= U_1(H^*, kC^*) - \lambda p_h = 0 \\
\frac{\partial L}{\partial C} &= U_2(H^*, kC^*)k - \lambda p_c = 0 \\
\frac{\partial L}{\partial \lambda} &= -p_h H^* - p_c C^* + y = 0
\end{aligned}$$

b. Differentiating the first order conditions by k we get

$$\begin{aligned}
U_{11}\frac{\partial H^*}{\partial k} + U_{12}\left(C^* + k\frac{\partial C^*}{\partial k}\right) - \frac{\partial \lambda^*}{\partial k}p_h &= 0 \\
U_{21}k\frac{\partial H^*}{\partial k} + U_2 + U_{22}k\left(C^* + k\frac{\partial C^*}{\partial k}\right) - \frac{\partial \lambda^*}{\partial k}p_c &= 0 \\
-p_h\frac{\partial H^*}{\partial k} - p_c\frac{\partial C^*}{\partial k} &= 0
\end{aligned}$$

By Crammer's rule,

$$\frac{\partial C^*}{\partial k} = \frac{\begin{vmatrix} U_{11} & -U_{12}C^* & -p_h \\ U_{21}k & -U_2 - U_{22}kC^* & -p_c \\ -p_h & 0 & 0 \end{vmatrix}}{\begin{vmatrix} U_{11} & kU_{12} & -p_h \\ U_{21}k & U_{22}k^2 & -p_c \\ -p_h & -p_c & 0 \end{vmatrix}}$$

At the optimal levels of H and C (i.e., H^*, C^*) the denominator is positive due to quasiconcavity of U. We assume it is strictly positive.

The numerator can be written as follows

$$\begin{aligned}
& -p_h\left(p_c C^* U_{12} - p_h U_2 - kC^* p_h U_{22}\right) \\
= & -p_h^2(\underbrace{\frac{p_c}{p_h}}_{\frac{kU_2}{U_1}\text{(from FOC)}} C^* U_{12} - U_2 - kC^* U_{22}) \\
= & -p_h^2\left(\frac{kU_2}{U_1}C^* U_{12} - kC^* U_{22} - U_2\right) \\
= & \; p_h^2\left(\frac{kC^*}{U_1}\underbrace{(U_1 U_{22} - U_2 U_{12})}_{\geq 0 \text{ if the housing is interior}} + U_2\right)
\end{aligned}$$

Hence, a sufficient condition for $\frac{\partial C^*}{\partial k} \geq 0$ is the inferiority of Housing.

2. The Lagrangian for the problem is given by

$$L = \sum\nolimits_{i=1}^{n} u_i(x_i) - \lambda \left(\sum\nolimits_{i=1}^{n} x_i p_i - y \right).$$

The first order conditions are given by

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= u_i'(x_i^*) - \lambda^* p_i = 0 \text{ for } i = 1, 2, .., n & (1) \\ \frac{\partial L}{\partial \lambda} &= -\sum\nolimits_{i=1}^{n} x_i^* p_i + y = 0 & (2) \end{aligned}$$

Differentiate (1) with respect to y

$$u_i''(x_i^*) \frac{\partial x_i^*}{\partial y} - \frac{\partial \lambda^*}{\partial y} p_i = 0$$

Then

$$\frac{\partial x_i^*}{\partial y} = \frac{\frac{\partial \lambda^*}{\partial y} p_i}{u_i''(x_i^*)} \text{ for } i = 1, 2, ..., n$$

Since, by assumption, $u_i''(x_i^*) \leq 0$ and $p_i \geq 0$ for all $i = 1, 2, ..., n$, then the sign of $\frac{\partial \lambda^*}{\partial y}$ defines the sign of $\frac{\partial x_i^*}{\partial y}$ for all $i = 1, 2, ..., n$.

It follows that if one good is inferior, then all of them are inferior. This is not possible since u is strictly increasing.

Then we conclude that all goods are normal.

3. We know that $u(x)$ is linearly homogenous.

a. Note that

$$\begin{aligned} e(p, \widetilde{u}) &= \min\nolimits_{x \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^{n} p_i x_i : u(x) \geq \widetilde{u} \right\} \\ &= \min\nolimits_{x \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^{n} p_i \widetilde{u} \left(\frac{x_i}{\widetilde{u}} \right) : \frac{u(x)}{\widetilde{u}} \geq 1 \right\} \\ &= \min\nolimits_{x \in \mathbb{R}_+^n} \left\{ \widetilde{u} \sum_{i=1}^{n} p_i \left(\frac{x_i}{\widetilde{u}} \right) : u \left(\frac{x}{\widetilde{u}} \right) \geq 1 \right\} \end{aligned}$$

Define $z = \frac{x}{\widetilde{u}}$, then

$$\begin{aligned} e(p, \widetilde{u}) &= \widetilde{u} \min_{z \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^{n} p_i (z_i) : u(z) \geq 1 \right\} \\ &= \widetilde{u} e(p, 1). \end{aligned}$$

b. By duality, $e(p, v(p, y)) = y$. Then,

$$e(p, v(p, y)) = v(p, y)e(p, 1) = y.$$

This means that

$$v(p, y) = y\frac{1}{e(p, 1)}.$$

Differentiating with respect to y,

$$\frac{\partial v(p, y)}{\partial y} = \frac{1}{e(p, 1)}$$

which is independent of $\widetilde{u}$.

4. We know that $\begin{pmatrix} \frac{\partial x_1^h}{\partial p_1} & \frac{\partial x_1^h}{\partial p_2} \\ \frac{\partial x_2^h}{\partial p_1} & \frac{\partial x_2^h}{\partial p_2} \end{pmatrix} = \begin{pmatrix} a & b \\ 2 & -\frac{1}{2} \end{pmatrix}$.

Since this matrix is symmetric, $b = 2$.
In addition, $x_1^h(p, u)$ and $x_2^h(p, u)$ are homogenous of degree zero in p.
Then, by Euler Theorem,

$$\begin{aligned} \frac{\partial x_1^h(p, u)}{\partial p_1} p_1 + \frac{\partial x_1^h(p, u)}{\partial p_2} p_2 &= 0 \\ a8 + 2p_2 &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial x_2^h(p, u)}{\partial p_1} p_1 + \frac{\partial x_2^h(p, u)}{\partial p_2} p_2 &= 0 \\ 16 - \left(\frac{1}{2} p_2\right) &= 0 \\ p_2 &= 32 \end{aligned}$$

Thus, $a = -8$.