Solution to Problem Set 2

1. The problem is

$$\max_{x_1,x_2}\{\min\{x_1,\alpha x_2\} : p_1x_1 + p_2x_2 \le y\}.$$

At the optimal (x_1^*, x_2^*) we should have $x_1^* = \alpha x_2^*$ and $p_1x_1^* + p_2x_2^* = y$. Then,

$$p_1\alpha x_2^* + p_2x_2^* = (p_2 + \alpha p_1)\, x_2^* = y.$$

It follows that

$$x_2^* = \frac{y}{(p_2 + \alpha p_1)} \text{ and } x_1^* = \frac{\alpha y}{(p_2 + \alpha p_1)}.$$

2. The maximization problem for m periods is

$$\max_{(x_t)_{t=0}^m}\{\sum_{t=0}^m \beta^t \ln(x_t) : \sum_{t=0}^m x_t = 1\}.$$

The Lagrangian for this problem is given by

$$L = \sum_{t=0}^m \beta^t \ln(x_t) - \lambda\left(\sum_{t=0}^m x_t - 1\right)$$

The FOC are

$$\frac{\partial L}{\partial x_t} = \beta^t \frac{1}{x_t^*} - \lambda^* = 0 \text{ for all } t = 0, 1, ..., m \quad (1)$$

$$\frac{\partial L}{\partial \lambda} = -\sum_{t=0}^m x_t^* + 1 = 0 \quad (2)$$

From (1) we have

$$\beta^t/\lambda^* = x_t^*.$$

From (2) we have

$$\sum_{t=0}^m x_t^* = 1.$$

Combining these to conditions we get

$$\sum_{t=0}^m \beta^t/\lambda^* = 1.$$

Then,

$$\lambda^* = \frac{1-\beta^{m+1}}{1-\beta} \quad \text{and} \quad x_t^* = \frac{\beta^t(1-\beta)}{1-\beta^{m+1}}.$$

Notice that if $m \to \infty$, then

$$x_t^* = \beta^t(1-\beta) \text{ for all } t = 0, 1, ...$$

3. Let us consider

$$u\left(x_1, x_2\right) = x_1 + x_2$$

Then,

$$\left(x_1^*, x_2^*\right) = \begin{cases} \left(0, \frac{y}{p_2}\right) & \text{if} \quad p_1 > p_2 \\ p_1 x_1^* + p_2 x_2^* = y \text{ with } x_1^*, x_2^* \geq 0 & \text{if} \quad p_1 = p_2 \\ \left(\frac{y}{p_1}, 0\right) & \text{if} \quad p_1 < p_2 \end{cases}$$

It follows that

$$V(p_1, p_2, y) = \frac{y}{\min\{p_1, p_2\}}.$$

By duality, we know that

$$V(p_1, p_2, e(p_1, p_2, u)) = u.$$

Then,

$$e(p_1, p_2, u) = u \min\{p_1, p_2\}.$$

4. Let us consider

$$u\left(x_1, x_2\right) = \begin{cases} -\infty & \text{if } x_1 < 1 \text{ and/or } \; x_2 < 3 \\ \left(x_1 - 1\right)\left(x_2 - 3\right) & \text{otherwise} \end{cases}$$

The Lagrangian for this problem (assuming $y \geq p_1 + 3p_2$) is given by

$$L = (x_1 - 1)(x_2 - 3) - \lambda(p_1 x_1 + p_2 x_2 - y)$$

The FOC are

$$\frac{\partial L}{\partial x_1} = (x_2^* - 3) - \lambda^* p_1 = 0 \tag{3}$$

$$\frac{\partial L}{\partial x_2} = (x_2^* - 1) - \lambda^* p_2 = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda} = -p_1 x_1^* - p_2 x_2^* + y = 0 \tag{5}$$

From (3) and (4) we have that

$$\frac{(x_2^* - 3)}{p_1} = \frac{(x_1^* - 1)}{p_2}.$$

Then,

$$p_1 x_1^* - p_1 = p_2 x_2^* - 3p_2$$

Substituting these expressions in (5) we get

$$p_1 x_1^* + p_1 x_1^* - p_1 + 3p_2 = y.$$

Then,

$$x_1^* = \frac{y - 3p_2 + p_1}{2p_1} \text{ and } x_2^* = \frac{y - p_1 + 3p_2}{2p_2} \quad \text{if } y \geq p_1 + 3p_2$$

By plugging (x_1^*, x_2^*) in $u(x_1, x_2)$ we get

$$V(p_1, p_2, y) = \begin{cases} -\infty & \text{if } y < p_1 + 3p_2 \\ \left(\frac{y - p_1 - 3p_2}{2p_1}\right)\left(\frac{y - p_1 - 3p_2}{2p_2}\right) & \text{if } y \geq p_1 + 3p_2 \end{cases}$$

Finally, by duality, $V(p_1, p_2, e(p_1, p_2, u)) = u$. Then,

$$e(p_1, p_2, u) = 2(up_1p_2)^{1/2} + p_1 + 3p_2.$$

5. We know that $u(x)$ is differentiable and strictly quasiconcave and $x(p, y)$ is differentiable.

a. We know $u(x)$ is homogenous of degree 1.

- We need to show that $V(p, y)$ is homogenous of degree 1 in y, i.e., $V(p, ky) = kV(p, y)$ for all $k \geq 0$.

Since $u(x)$ is homogenous of degree 1

$$\begin{aligned} V(p, ky) &= \max_x \{u(x) : x \cdot p = ky\} \\ &= \max_x \left\{u\left(k\frac{x}{k}\right) : \frac{x}{k} \cdot p = y\right\}. \end{aligned}$$

Let us define $x/k = z$. Then,

$$\begin{aligned} V(p, ky) &= \max_z \{k.u(z) : z \cdot p = y\} \\ &= k \max_z \{u(z) : z \cdot p = y\} = kV(p, y) \end{aligned}$$

which completes the proof.

- We need to show that $x(p, y)$ is homogenous of degree 1 in y, i.e., $x(p, ky) = kx(p, y)$ for all $k \geq 0$.

Let us define $x/k = z$. Since $u(x)$ is homogenous of degree 1

$$\begin{aligned} x(p, ky) &= \arg\max_x \{u(x) : x \cdot p = ky\} \\ &= \arg\max_x \{u\left(k\frac{x}{k}\right) : \frac{x}{k} \cdot p = y\} \\ &= k \arg\max_z \{ku(z) : z \cdot p = y\} \\ &= k \arg\max_z \{u(z) : z \cdot p = y\} = kx(p, y). \end{aligned}$$