Solution to Problem Set 1

1. For each function below, determine whether it is concave, quasiconcave, or neither. (Assume $x, y \geq 0$.)

a. $f(x, y) = x\sqrt{y}$

- Concavity

$H(f(x)) = \begin{pmatrix} 0 & \frac{1}{2}\frac{1}{\sqrt{y}} \\ \frac{1}{2}\frac{1}{\sqrt{y}} & -\frac{1}{4}y^{-\frac{3}{4}}x \end{pmatrix}$

Since $|H(f(x))| = -\frac{1}{4}\frac{1}{y} < 0$ and f is C^2, then f is not concave.

- Quasiconcavity

Let $f(x, y) = e^{\ln x + \frac{1}{2}\ln y} = e^{g(x,y)}$.
Note that $g(x, y) = \ln x + \frac{1}{2}\ln y$ is a concave function.
Since $f(x, y)$ is a monotone transformation of $g(x, y)$, then $f(x, y)$ is quasiconcave.

b. $f(x, y) = u(x) + v(y)$, with $u(x)'' \leq 0$ and $v''(y) \leq 0$ for all $x, y \geq 0$.

$H(f(x, y)) = \begin{pmatrix} u''(x) & 0 \\ 0 & v''(y) \end{pmatrix}$

We know that $u''(x) \leq 0$ and $v''(y) \leq 0$. In addition,

$$|H(f(x, y))| = u''(x)v''(y) \geq 0.$$

Then f is concave —and also quasiconcave.

c. $f(x) = 2x - (x+1)^{-1} + (x+1)^{-2}$, $x > 0$

$$f''(x) = \begin{cases} \geq 0 \text{ if } 0 < x \leq 2 \\ < 0 \text{ if } x > 2 \end{cases}$$

Then f is not concave. However, since $f'(x) \geq 0 \ \forall x > 0$, then f is quasiconcave.

2. $h(x) = x^3 + x, \qquad g(x) = -2x, \qquad x \in \mathbb{R}$

- h is increasing, then it is quasiconcave.
- g is decreasing, so it is quasiconcave.
- We next show that $h(x) + g(x) = x^3 - x$ is not quasiconcave. Let $m(x) = x^3 - x$ and consider two points: $x = 1$ and $x' = -\frac{1}{2}$.

1. • Notice $x'' = \frac{2}{3}x + \frac{1}{3}x' = \frac{1}{2}$ is a convex combination of x and x'. However,

$$m\left(\frac{1}{2}\right) = -\frac{3}{8} < \min\left\{m(1), m\left(-\frac{1}{2}\right)\right\} = 0.$$

Then, function $m(x)$ is not quasiconcave

3. $f : \mathbb{R}^n \to \mathbb{R}$ is concave, $g : \mathbb{R} \to \mathbb{R}$ is increasing and concave. We need to show that $h(x) = g(f(x))$ is concave.

Let $x, x' \in \mathbb{R}^n$ and consider $\lambda \in [0, 1]$. Then

$$\begin{aligned} h(\lambda x + (1-\lambda)x') &= \underbrace{g(f(\lambda x + (1-\lambda)x') \geq g(\lambda f(x) + (1-\lambda) f(x'))}_{\text{by concavity of } f \text{ and the fact that } g \text{ is increasing}} \\ &\underbrace{\geq \lambda g(f(x)) + (1-\lambda) g(f(x'))}_{\text{by concavity of } g} = \lambda h(x) + (1-\lambda) h(x') \end{aligned}$$

The result follows since λ was arbitrarily selected.

4. We know that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $u : X \to \mathbb{R}$ represents $\succeq$.

We need to show that $v(x) = f(u(x))$ is also a utility function that represents $\succeq$.

Dy definition, "u represents $\succeq$" means that $\forall x, y \in X$

$$x \succeq y \Longleftrightarrow u(x) \geq u(y).$$

The result follows because f is strictly increasing and thereby f^{-1} is also strictly increasing. Since f is strictly increasing

$$u(x) \geq u(y) \Rightarrow f(u(x)) \geq f(u(y)).$$

Since f^{-1} is also strictly increasing

$$f(u(x)) \geq f(u(y)) \Rightarrow u(x) \geq u(y).$$

Then,

$$v(x) = f(u(x)) \geq f(u(y)) = v(y) \Leftrightarrow u(x) \geq u(y) \forall x, y \in X.$$

5. We need to show that $f(x, y) = \min\{x, y\}$ is a concave function.

Let $(x, y), (x', y') \in \mathbb{R}^2$ and $\lambda \in [0, 1]$. We need to prove that

$$\min\{\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y'\} \geq \lambda \min\{x, y\} + (1-\lambda)\min\{x', y'\}. \quad (1)$$

There are 4 cases to consider

(a) $x \geq y$ and $x' \geq y'$

(b) $x \leq y$ and $x' \leq y'$

(c) $x > y$ and $x' < y'$

(d) $x < y$ and $x' > y'$

In fact, by symmetry, we only need to consider two cases. For instance, (a) and (c).

Let us consider first case (a). Then,

$$\begin{aligned} \min\{\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y'\} &= \lambda y + (1-\lambda)y' \\ \lambda \min\{x, y\} + (1-\lambda)\min\{x', y'\} &= \lambda y + (1-\lambda)y'. \end{aligned}$$

Then (1) holds.

Let us consider next case (c). Assume (without loss of generality) that

$$\lambda x + (1-\lambda)x' \geq \lambda y + (1-\lambda)y'.$$

Then,

$$\min\{\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y'\} = \lambda y + (1-\lambda)y'.$$

In addition,

$$\min\{x, y\} = y \text{ and } \min\{x', y'\} = x'$$

The rest of the proof follows by contradiction. Then, assume

$$\lambda y + (1-\lambda)y' < \lambda y + (1-\lambda)x'.$$

This implies that $x' > y'$, which contradicts the fact that $x' < y'$.