Solution for Midterm

Question 1

(a) The Lagrangian for the consumer problem is given by

$$L = U(x_1, x_2) - \lambda (p_1 x_1 + p_2 x_2 - y).$$

Thus, by the Envelop Theorem

$$\partial V(p_1, p_2, y) / \partial y = \lambda^* \geq 0.$$

(b) Notice that, by part a, $\partial^2 V(p_1, p_2, y) / \partial y^2 = \partial \lambda^* / \partial y$.

The FOCs for the consumer problem are given by

$$\begin{aligned} \partial L / \partial x_1 &= U_1 - \lambda^* p_1 = 0 \\ \partial L / \partial x_2 &= U_2 - \lambda^* p_2 = 0 \\ \partial L / \partial \lambda &= -p_1 x_1^* - p_2 x_2^* + y = 0 \end{aligned}$$

Differentiating the FOC with respect to y, we get

$$\begin{aligned} U_{11} \frac{\partial x_1^*}{\partial y} + U_{12} \frac{\partial x_2^*}{\partial y} - p_1 \frac{\partial \lambda^*}{\partial y} &= 0 \\ U_{21} \frac{\partial x_1^*}{\partial y} + U_{22} \frac{\partial x_2^*}{\partial y} - p_2 \frac{\partial \lambda^*}{\partial y} &= 0 \\ -p_1 \frac{\partial x_1^*}{\partial y} - p_2 \frac{\partial x_2^*}{\partial y} + 1 &= 0 \end{aligned}$$

By Crammer's rule we get that

$$\frac{\partial \lambda^*}{\partial y} = \frac{\begin{vmatrix} U_{11} & U_{12} & 0 \\ U_{21} & U_{22} & 0 \\ -p_1 & -p_2 & -1 \end{vmatrix}}{\begin{vmatrix} U_{11} & U_{12} & -p_1 \\ U_{21} & U_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix}}$$

The denominator is weakly positive by the SOC, and we assume it is different from 0. The numerator is negative if

$$U_{11} U_{22} - U_{12} U_{21} \geq 0.$$

Thus, a sufficient condition for $\partial^2 V(p_1, p_2, y) / \partial y^2 = \partial \lambda^* / \partial y \leq 0$ is concavity of $U(x_1, x_2)$.

Question 2

(a) The problem of the consumer is given by

$$\max_{x_1, x_2} \{\min \{x_1 - x_2, 3x_2 - x_1\} : x_1 + x_2 = 300\}.$$

At the optimal solution

$$x_1 - x_2 = 3x_2 - x_1.$$

Thus, $x_1^* = 2x_2^*$. Substituting in the budget restriction, we get $2x_2^* + x_2^* = 300$. Thus,

$$x_1^* = 200 \text{ and } x_2^* = 100.$$

(b) The problem of the consumer is given by

$$\min_{x_1, x_2} \{x_1 + x_2 : \min\{x_1 - x_2, x_2 - 3x_1\} = -10\}.$$

At the optimal solution

$$\begin{aligned} x_1 - x_2 &= -10 \\ x_2 - 3x_1 &= -10. \end{aligned}$$

Thus, $x_1 + 10 - 3x_1 = -10$. Thus, $x_1^* = 10$ and $x_2^* = 20$.

Question 3

(a) The problem of the firm is

$$\Pi(p, w, r) = \max_{L,K} \{pf(L, K) - wL - rK\}.$$

By the Envelop Theorem

$$\partial\Pi(p, w, r)/\partial r = -K^*.$$

Thus,

$$-\partial^2\Pi(p, w, r)/\partial r\partial w = \partial K^*/\partial w = -\partial^2\Pi(p, w, r)/\partial w\partial r = \partial L^*/\partial r < 0$$

where the second equality follows by Young's theorem and the inequality by the statement in the problem. Thus, L^* decreases in r.

(b) The problem of the firm can be written as

$$\Pi(p, w, r) = \max_{y} \{py - C(w, r, y)\}.$$

By the Envelop Theorem

$$\partial\Pi(p, w, r)/\partial p = y^*$$

Differentiating with respect to p we get

$$\partial^2\Pi(p, w, r)/\partial p^2 = \partial y^*/\partial p \geq 0$$

since the profit function is convex in p.